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A SECOND-ORDER SATELLITE ORBIT THEORY, WITH COMPACT RESULTS IN CYLINDRICAL COORDINATES

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First-order perturbations of a low-eccentricity satellite orbit due to the general harmonic coefficient J_{lm} of the Earth's gravitational field are derived, and compactly expressed (see equations (92)–(94)) in cylindrical polar coordinates. For the dominant harmonic J_2, the perturbations are taken to second order, and it is shown how formulae for the second-order variation of the orbital elements depend on the definition of the mean elements used for reference. With a particular choice of mean elements, the formulae for perturbations in cylindrical coordinates are again very compact (see equations (297), (315) and (321)).

The general approach also yields first-order perturbations due to lunisolar gravity and eclipse-free solar radiation. The paper finishes with a set of untruncated expressions (valid for any eccentricity) for J_2^2 secular and long-periodic perturbations.

1. Introduction

1.1. *Previous papers*

The description of the motion of an artificial satellite in the axisymmetric field due to the low-degree zonal harmonics of the geopotential has been recognized as the 'main problem' in the theory of satellite orbits, and many solutions have been published. The first R.A.E. paper (King-Hele & Gilmore 1957) appeared in the same month as the first satellite and was succeeded by the theory which Merson (1961) developed as a basis for the first proper orbit determination program of the R.A.E. (Merson 1963 *a*). He took this theory as far as J_6 (the coefficient of the zonal harmonic

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of degree 6) and then (Merson 1963*b*) compared it with the theory of Kozai (1959), taken to J_4, that pioneered the orbit analysis of the Smithsonian Astrophysical Observatory. Kozai's paper may be regarded as one of the two classic English-language papers – his approach was the solution of Lagrange's planetary equations and it is summarized in the book by Roy (1978). The other classic paper was Brouwer's (1959), published at the same time as Kozai's; Brouwer's approach was the solution of Hamilton's equations in Delaunay variables, after the method of von Zeipel (1916, 1918), and an account is to be found in the book by Brouwer & Clemence (1961). One of the first textbooks to give results (for J_2 only) was that of Sterne (1960). In a second theory, Merson (1966) adopted the Kozai approach, extending it to cover any number of zonal harmonics, and Gooding (1966) also obtained results for an arbitrary J_l. This second theory of Merson was developed as the basis for a new orbit determination program, PROP (Gooding & Tayler 1968), that was later improved (Gooding 1974) and is still in use.

The early papers normally provided long-term solutions of the 'main problem', i.e. gave formulae for perturbations in mean orbital elements, to $O(J_2^2)$, where $J_l = O(J_2^2)$ if $l > 2$, but only took short-periodic perturbations to $O(J_2)$, neglecting them entirely for higher-order harmonics. It was found by Vinti (1959, 1961, 1963, 1966) that, by use of spheroidal coordinates, a complete solution could be found for a field including J_2 and J_3, but then the higher harmonics had to be (rational) functions of J_2 and J_3, with $J_4 = (J_3^2 - J_2^3)/J_2$, $J_5 = J_3(J_3^2 - 2J_2^3)/J_2^2$ and (in general) $J_l = \sigma^l \sin{(l-1)}\,\gamma/\sin\gamma$, where $\sigma = J_2^{\frac{1}{2}}$ and $\cos\gamma = J_3/2J_2^{\frac{3}{2}}$, so the general problem remained. (The complexity of this expression results from the continued use of a geopotential expanded in spherical coordinates.)

The first paper to take short-periodic perturbations to $O(J_2^2)$, with J_3 and J_4 covered, seems to have been that of Petty & Breakwell (1960). Kozai (1962), in a comprehensive development to $O(J_2^2)$ for short-periodic perturbations and $O(J_2^3)$ for long-term perturbations, with J_5, J_6, J_7 and J_8 covered but regarded as $O(J_2^3)$, changed his approach to the method of von Zeipel that had been used by Brouwer. Aksnes (1970) followed Hori (1966) in basing his approach on the use of Lie series; by reference to a suitable intermediate orbit, he thereby obtained results (to J_4 only) that were essentially equivalent to Kozai's but complete, much more compact and (unlike Kozai's) valid for zero eccentricity. More elementary (and hence more comprehensible) methods, just based on the planetary equations of Lagrange, were adopted in a series of French papers; J_2^2 short-periodic perturbations were given by Bretagnon (1972) and J_2^3 long-term effects by Berger (1972), none of the higher harmonics being covered.

Solution of the J_2 problem to yet a further order – short-periodic perturbations to $O(J_2^3)$ and long-term effects to $O(J_2^4)$ – was achieved by Deprit & Rom (1970) and again by Kutuzov (1976), in both cases by the application of computer algebra to a method based on Lie series. These results cover J_2 only, however, and are only valid for moderately small values of the orbital eccentricity, e, since they are expressed as truncated power series in e. Techniques based on computer algebra are obviously very powerful, and two papers have recently been published that give main-problem solutions involving vast numbers of terms. Berger & Walch (1977) cover harmonics as far as J_7; they only go to second order in short-periodic perturbations and (basically) third order in long-term effects, but here third-order covers terms such as secular terms in $J_5 J_7/J_2$ that come from the treatment of fourth-order coupling between J_5 and J_7, so that many combinations occur. Kinoshita (1977) covers only the classical main-term harmonics, J_2, J_3 and J_4, but goes to third order for short-periodic perturbations and to fourth order for long-term effects; he adopts the perturbation method of Hori (1966) and, for satellite motion in a low-eccentricity

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orbit (as evaluated by comparisons with numerically integrated ephemerides), claims an accuracy better than 1 cm over a month.

In the papers so far referred to, attention was restricted to the zonal harmonics, but an excellent early paper by Groves (1960) gave general formulae that also covered the tesseral harmonics – a term now usually taken to include the sectorial harmonics. The standard reference, however, specifying the effects of the general spherical harmonic coefficient, J_{lm}, is the textbook of Kaula (1966). Mention should also be made of the paper by Cook (1963) that gives a very comprehensive review of the various general approaches to the main problem and of the early individual papers.

Much of the literature cited is extremely difficult to follow, even for a reader with considerable experience of orbital analysis, mainly because of the extreme sophistication of the methods used, but probably also because of unfamiliarity with the notation. Without understanding the methods it is difficult to interpret the results – to pick out (say) a dominant set of terms for $O(J_2^2)$ perturbations for a near-circular orbit becomes virtually impossible. This is particularly true if expansions are made in terms of the quantity $(1-e^2)^{\frac{1}{2}}$, as in Brouwer (1959), Kozai (1962) and Kinoshita (1977), in each of which papers this quantity is denoted by η, and also effectively in Deprit & Rom (1970) – the notation for $(1-e^2)^{\frac{1}{2}}$ is q in the present paper (λ is also often used in celestial mechanics), but expansions in q are avoided.

1.2. *The present paper*

In many applications, involving satellites in orbits of low eccentricity, there are significant perturbations of order J_{lm} (assumed to cover the J_l with $l > 2$) and J_2^2, including terms of short period, but terms in $J_{lm}e$ and J_2^2e are entirely negligible. The significant terms may be obtained quite easily, by starting with a solution of Lagrange's planetary equations that is first-order complete in J_2 and general (but truncated) in J_{lm}, and then using the solution to J_2e to build up the J_2^2 terms. Furthermore, when an appropriate set of non-singular elements is used, the J_2^2 results can be expressed very compactly. This is the approach of the present paper; the particular non-singular elements are (as defined in § 2.1) a, i, Ω, ξ, η and U, and first-order short-periodic perturbations in these elements, as given by equations (148)–(153), lead to the second-order perturbations given in § 7.2–7.6 and 7.9.

This paper makes two principal contributions to the solution of the 'main problem', and discusses various aspects of the philosophy of perturbation theory. The first contribution concerns the derivation of formulae for perturbations in position, not merely orbital elements. If the satellite's position can be expressed by three suitable coordinates, then the number of perturbation formulae required is immediately halved. (In certain applications, for example the processing of Doppler data, formulae for three velocity components will also be required, but these can be obtained from the position formulae by differentiation.) The reduction will be greater if the formulae for position are simpler than those for certain of the elements, as is clear from Kozai's original paper (1959). By introducing r (geocentric distance) and u (argument of latitude), Kozai was able to reduce long expressions for the (first-order) short-periodic perturbations in the elements a, e, ω and M into compact expressions for the perturbations in r and u; the expressions for the perturbations in i and Ω are much shorter and he left these alone. In developing the Kozai approach for use with PROP, Merson (1966) presented six perturbation formulae that are very compact and completely free of singularity, and Gooding (1974) modified them slightly to reduce nonlinear effects.

However, the most natural development of the Kozai approach is by adding a third coordinate

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to r and u, or rather to quantities r' and u' differing only slightly from r and u. The third coordinate is the cross-track displacement c, and the combination of r', u' and c amounts to a set of (geocentric) cylindrical polar coordinates defined relative to a steadily rotating mean orbital plane. The formulae for short-periodic perturbations in these coordinates constitute the main results of this paper.

The paper's other contribution to the main problem relates to the definition of 'mean' orbital elements, and hence to the 'mean orbital plane' just referred to. The concept of a mean element underlies all approaches to the main problem, but there is an ambiguity in every such element in that the short-periodic perturbation (which, added to the mean element, gives the uniquely defined osculating element) is arbitrary by any quantity that is *free* of short-periodic perturbation. Thus if ζ stands for one of the orbital elements and

$$\zeta_{\mathrm{osc}} = \bar{\zeta} + \delta\zeta_{\mathrm{s.p.}},$$

then $\delta\zeta_{\mathrm{s.p.}}$ is a function of u, i.e. of orbital position, but is arbitrary to the extent of a quantity that is a function of the mean elements.

Most papers on the main problem do not explain how their mean elements are defined, nor indeed admit that there is any problem. This makes it difficult to compare the results of different theories and to interpret the published elements of actual satellites. The present paper tackles the problem head-on, for the first- and second-order effects of J_2, by introducing arbitrary 'constants', k_ζ and $k_{2\zeta}$, into formulae, so that general results can be given without a pre-empted interpretation of particular mean elements. First-order such k_ζ are introduced in § 6.2 (subsequent to further discussion of mean elements in § 3.1), the most important being k_a, k_i and k_n.

There are three possible criteria that can decide the choice of the various k_ζ and which (tacitly at least) are applied in the literature. First, they may be chosen to make the expressions for perturbations in position as simple as possible; this is the philosophy adopted here. Secondly, they may be chosen so that time averages of the short-periodic perturbations are strictly zero; this amounts to making the k_ζ zero if perturbations are expressed in terms of mean anomaly (M) rather than true anomaly (v). Thirdly, the k_ζ may be made zero when the perturbations are expressed in terms of v; higher-order formulae will then be simpler than with the second choice.

This brings us to another important point. Early formulations of the short-periodic perturbations (see, for example, Kozai 1959; Sterne 1960) were in terms of v (or u, where $u-v = \omega$, the argument of perigee) and hence expressible in closed form. Later papers, and in particular those based on computer algebra, switched to M, for reasons that are not entirely clear, the result being expressions that are necessarily truncations of power series in e. The present paper, even though it truncates before $O(J_{1\mathrm{m}}\,e)$ and $O(J_2^2 e)$ terms, uses v (or rather u); there is no obvious reason why, in terms of v, the main problem should not be solved to further order without a requirement for infinite series in e (but see also the remarks of Deprit & Rom (1970) and Hori & Kozai (1975) – the difficulty lies in evaluating $\int v\,\mathrm{d}M$).

Section 9 of this paper is devoted to complete expressions for the J_2^2 long-term variation of the orbital elements, divided naturally into secular and long-periodic perturbations. When $O(J_2^2 e)$ effects can be neglected, no long-periodic perturbations arise (the meaning of 'second-order' in connection with these perturbations is discussed), but (as already indicated) it makes sense to take long-term perturbations to a higher order than short-periodic perturbations, unless the satellite's motion is only to be represented for a matter of, say, at most a revolution or two.

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1.3. *Notation*

a	semi-major axis
a'	constant 'mean a' given by energy integral for zonal harmonics
A, B, C	direction cosines of Sun or Moon
$\mathrm{A}_{\mathrm{l}}^{\mathrm{k}}(i)$	normalized inclination function
$\hat{A}, \hat{D}$	quantities related to $\mathrm{A}_{\mathrm{l}}^{\mathrm{k}}(i)$
c	excursion from mean orbital plane
c, s	$\cos i$, $\sin i$ in §8.3
$C_{\mathrm{nm}}, S_{\mathrm{nm}}$	spherical-harmonic coefficients
$\bar{C}_{\mathrm{nm}}, \bar{S}_{\mathrm{nm}}$	normalized versions of C_{nm} and S_{nm}
$\mathrm{C}_j, \mathrm{S}_j$	$\cos(jv+2\omega)$, $\sin(jv+2\omega)$
$\mathrm{D}\dot{\zeta}$	second-order component of $\dot{\zeta}$, made up of $\mathrm{D}_a\dot{\zeta}$, $\mathrm{D}_e\dot{\zeta}$, etc.
e	eccentricity
E	eccentric anomaly
f	$\sin^2 i$
$F_{\mathrm{lmp}}(i)$	inclination function
$\bar{F}_{\mathrm{lm}}^{\mathrm{k}}(i)$	normalized inclination function
$\hat{F}, \hat{F}'$	quantities related to $\bar{F}_{\mathrm{lm}}^{\mathrm{k}}(\mathrm{i})$
h	$1-\frac{3}{2}f$
i	inclination
i	$(-1)^{\frac{1}{2}}$
$I_{\mathrm{c}}, I_{\mathrm{s}}, II_{\mathrm{s}}$	definite integrals of $\cos 2\varpi$, $\sin 2\varpi$, I_{s}
$J_{\mathrm{nm}}, \bar{J}_{\mathrm{nm}}$	un-normalized and normalized spherical-harmonic coefficients
J_{n}	zonal-harmonic coefficient
$J_j(je)$	Bessel function
k_ζ	integration 'constant' associated with element ζ
$k_{2\zeta}$	second-order 'constant' associated with element ζ
k, l, m, p	indices associated with U and F functions
K	$\frac{3}{2}J_2(R/p)^2$
$\bar{K}$	mean K
l	first-order-definable quantity related to u'
L	first-order-definable quantity such that $\dot{L} = \dot{M}+q\dot{\psi}$
M	mean anomaly
n	mean motion
N_{lm}	normalizing factor
p	parameter (semi-latus rectum) of ellipse
$P_{\mathrm{l}}^{\mathrm{m}}(\sin\beta)$	Legendre-polynomial function of $\sin\beta$
q	$(1-e^2)^{\frac{1}{2}}$
r	geocentric radius vector
r'	quantity closely related to r
r', u', c	set of cylindrical polar coordinates
R	Earth's equatorial radius or distance of external body
$\mathrm{R}_1, \mathrm{R}_2, \mathrm{R}_3$	rotation operators (matrices)
s	ratio of rates: Earth's rotation to satellite's nodal revolution

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t	time
T	affix for matrix transposition
u	argument of latitude
u'	quantity closely related to u
U	sum of M and ω
$\mathrm{U_{nm}}$	representative term of gravitational potential
$\mathrm{U^k_{nm}}$	component of $\mathrm{U_{nm}}$
v	true anomaly
v_M	$v-M$
$V(v, e)$	function of v and e given by $-\frac{4}{3}e^{-2}(v-M-2e\sin v)$
x, y, z	geocentric inertial coordinates
α	$a^{\frac{1}{2}}$
β	geocentric latitude
γ	$\frac{1}{2}\pi - u$ (except in § 1.1)
$\delta\zeta$	short-periodic perturbation in ζ
$\Delta\zeta$	long-periodic component of variation in ζ
ζ	representative orbital element
$\bar{\zeta}$	mean ζ
ζ_1, ζ_2	quantities such that $\bar{K}\zeta_1$ and $\bar{K}^2\zeta_2$ give first- and second-order short-periodic perturbations
η	$e\sin\omega$
θ, ϕ	angles such that $(A, B, C) = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$
ι	$i-\bar{i}$ in § 8.3
λ	longitude
λ_{nm}	longitude phase associated with J_{nm}
μ	Earth's gravitational constant
μ_{d}	gravitational constant for a disturbing body
ν	sidereal angle
ξ	$e\cos\omega$
ρ	first-order-definable quantity such that $\dot{L} = \dot{\rho} + n$
σ	modified mean anomaly at epoch (except in § 1.1)
υ	$u'-u$
χ	$\frac{1}{2}\pi - \Omega + \nu + \lambda_{\mathrm{nm}}$
ψ	first-order-definable quantity such that $\dot{\psi} = \dot{\omega} + \dot{\Omega}\cos i$
ω	argument of perigee
Ω	right ascension of the ascending node

2. Background

2.1. *Osculating elements*

The standard osculating, i.e. instantaneously defined, elements of an elliptic orbit are a (semi-major axis), e (eccentricity), i (inclination), Ω (right ascension of the node), ω (argument of perigee) and M (mean anomaly). Since e is usually assumed to be small, we shall also employ the non-singular elements ξ, η and U, defined by

$$\xi = e\cos\omega, \tag{1}$$

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$$\eta = e \sin \omega \tag{2}$$

and

$$U = M + \omega. \tag{3}$$

The mean motion, n, though defined by

$$n^2 a^3 = \mu, \tag{4}$$

where μ is the Earth's gravitational constant, will often be treated as an element in its own right. As indicated previously (Gooding 1966), the first-order perturbation analysis is facilitated by introduction of some subsidiary quantities. Thus, we define $\dot{\psi}$, $\dot{L}$ and $\dot{\rho}$ by

$$\dot{\psi} = \dot{\omega} + \dot{\Omega} \cos i, \tag{5}$$

$$\dot{L} = \dot{M} + (1-e^2)^{\frac{1}{2}} \dot{\psi} \tag{6}$$

and

$$\dot{\rho} = \dot{L} - n. \tag{7}$$

Because i and e are not constant, proper integrated parameters ψ, L and ρ cannot be defined but this is of little moment – it is legitimate (and extremely useful) to be able to work with perturbations $\delta\psi$, δL and $\delta\rho$, such that

$$\delta\psi = \delta\omega + \delta\Omega \cos i, \tag{8}$$

$$\delta L = \delta M + (1-e^2)^{\frac{1}{2}} \delta\psi \tag{9}$$

and

$$\delta\rho = \delta L - \int \delta n \, \mathrm{d}t. \tag{10}$$

Because δM and $\delta\psi$ (though not δL) are in general $O(e^{-1})$, (3), (8) and (9) give

$$\delta L = \delta U + \delta\Omega \cos i + \tfrac{1}{2} e^2 \delta M + O(e^2). \tag{11}$$

We shall also use σ, the 'modified mean anomaly at epoch' (where $t = 0$), defined such that

$$M = \sigma + \int_0^t n \, \mathrm{d}t, \tag{12}$$

so that

$$\dot{M} = \dot{\sigma} + n. \tag{13}$$

For conciseness of formulae it is convenient to introduce quantities f, h, p and q, where

$$f = \sin^2 i, \tag{14}$$

$$h = 1 - \tfrac{3}{2} f, \tag{15}$$

$$p = a(1-e^2) \tag{16}$$

and

$$q = (1-e^2)^{\frac{1}{2}}. \tag{17}$$

Finally, we denote true anomaly and argument of latitude by v and u respectively, so that

$$u = \omega + v. \tag{18}$$

2.2. *Lagrange's planetary equations*

Rates of change of osculating elements may be expressed exactly by Lagrange's planetary equations (Roy 1978; Brouwer & Clemence 1961; Sterne 1960), namely

$$\dot{a} = \frac{2}{na} \frac{\partial U}{\partial M}, \tag{19}$$

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$$\dot{e} = \frac{1}{na^2e}\left\{q^2\frac{\partial U}{\partial M} - q\frac{\partial U}{\partial \omega}\right\}, \tag{20}$$

$$\dot{i} = \frac{\operatorname{cosec} i}{na^2q}\left\{\cos i\frac{\partial U}{\partial \omega} - \frac{\partial U}{\partial \Omega}\right\}, \tag{21}$$

$$\dot{\Omega} = \frac{\operatorname{cosec} i}{na^2q}\frac{\partial U}{\partial i}, \tag{22}$$

$$\dot{\omega} = \frac{1}{na^2}\left\{\frac{q}{e}\frac{\partial U}{\partial e} - \frac{\cot i}{q}\frac{\partial U}{\partial i}\right\} \tag{23}$$

and

$$\dot{\sigma} = -\frac{1}{na^2}\left\{\frac{q^2}{e}\frac{\partial U}{\partial e} + 2a\frac{\partial U}{\partial a}\right\}. \tag{24}$$

In (19)–(24), U is the potential (or disturbing function) assumed to be responsible for the variation of a, e, i, Ω, ω and σ. We also have – and this is the whole point of the definitions given by (5)–(7) –

$$\dot{\psi} = \frac{q}{na^2e}\frac{\partial U}{\partial e} \tag{25}$$

and

$$\dot{\rho} = -\frac{2}{na}\frac{\partial U}{\partial a}. \tag{26}$$

In first-order analysis in which $O(e)$ perturbations are ignored, we can express the Lagrangian equations in the following approximate form:

$$\dot{a} = \frac{2}{na}\frac{\partial U}{\partial v}, \tag{27}$$

$$\dot{e} = \frac{1}{na^2e}\left\{(1+2e\cos v)\frac{\partial U}{\partial v} - \frac{\partial U}{\partial \omega}\right\}, \tag{28}$$

$$\dot{i} = \frac{1}{na^2}\left\{\cot i\frac{\partial U}{\partial \omega} - \operatorname{cosec} i\frac{\partial U}{\partial \Omega}\right\}, \tag{29}$$

$$\dot{\Omega} = \frac{\operatorname{cosec} i}{na^2}\frac{\partial U}{\partial i}, \tag{30}$$

$$e\dot{\psi} = \frac{1}{na^2}\left\{\frac{\partial U}{\partial e} + 2\sin v\frac{\partial U}{\partial v}\right\} \tag{31}$$

and

$$\dot{\rho} = -\frac{2}{na}\frac{\partial U}{\partial a}. \tag{32}$$

Two points are worth making in connection with (31). First, $\partial U/\partial e$ does not mean the same thing as it does in (23)–(25), since it is now relative to v held constant, rather than M held constant. Secondly, the equation is presented with an e-factor, since this is how ψ affects satellite position; if we were concerned with near-equatorial orbits we would associate a factor $\sin i$ with $\dot{\Omega}$ in the same way. It is correct to keep e in the denominator of the right-hand side of (28), on the other hand; it always cancels out, since $\partial U/\partial v = \partial U/\partial \omega + O(e)$.

2.3. *Assumed potential*

We take the gravitational potential of the Earth in the usual form as a doubly infinite sum of spherical harmonics, of which the representative term may be expressed (Gooding 1966) as

$$U_{nm} = \frac{\mu}{r}\left(\frac{R}{r}\right)^{n} P_{l}^{m}(\sin\beta)\,(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda), \tag{33}$$

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where r, β (geocentric latitude) and λ (east longitude) are polar coordinates, R is the Earth's mean equatorial radius, and C_{nm} and S_{nm} are dimensionless coefficients. The indices l, m and n are integers with $n > 1$, $l = n$ and $0 \leqslant m \leqslant l$. Thus n is redundant, and does not normally appear in the expression. It is included here, however, to provide a useful generalization, in which (Gooding 1966) n can take negative values; we then have $l = -(n+1)$. The generalization in no way complicates the analysis, and permits the general formulae obtained to be applied at once to the derivation of perturbations due to solar radiation and lunisolar gravity (see § 5).

Instead of C_{nm} and S_{nm}, it is now customary to employ normalized equivalents, $\bar{C}_{nm}$ and $\bar{S}_{nm}$, defined by

$$C_{nm} = N_{lm}\bar{C}_{nm} \quad \text{and} \quad S_{nm} = N_{lm}\bar{S}_{nm}, \tag{34}$$

where N_{lm} is the reciprocal of the normalizing factor. Also, we may define 'polar equivalents', J_{nm}, $\bar{J}_{nm}$ and λ_{nm}, such that

$$C_{nm} = J_{nm}\cos\lambda_{nm} \quad \text{and} \quad S_{nm} = J_{nm}\sin\lambda_{nm}, \tag{35}$$

with the equivalent normalized relations. The standard value for N_{lm}, when $m \neq 0$, is given by

$$N_{lm}^2 = 2(2l+1)\,(l-m)!/(l+m)!, \tag{36}$$

and when $m = 0$, i.e. for *zonal* harmonics, by (36) without the initial factor of 2. However, it has traditionally been preferred, with the zonal harmonics, to use the coefficient J_n, introduced in § 1; J_n is defined as $-C_{n,0}$ (there being no $S_{n,0}$), so we can conveniently identify J_n with $\bar{J}_{n,0}$ by taking $N_{nm} = -1$ instead of the standard value.

2.4. *Formulae connecting mean and true anomaly*

Formulae connecting M and v will be required and are collected here for reference. They are valid for both osculating elements and the mean elements that are introduced in § 3.

The difference between M and v is expressed by the 'equation of the centre'. The following is an exact form of the equation, expressed as a double series of Bessel functions, as given by page 77 of Brouwer & Clemence (1961):

$$v = M + 2\sum_{j=1}^{\infty}\frac{\sin jM}{j}\left\{J_j(je) + \sum_{k=1}^{\infty}\left(\frac{e}{1+q}\right)^k[J_{j-k}(je) + J_{j+k}(je)]\right\}. \tag{37}$$

This leads to

$$v = M + 2e\sin M + \tfrac{5}{4}e^2\sin 2M - \tfrac{1}{12}e^3(3\sin M - 13\sin 3M) + O(e^4), \tag{38}$$

to which the equivalent v-series result is

$$M = v - 2e\sin v + \tfrac{3}{4}e^2\sin 2v - \tfrac{1}{3}e^3\sin 3v + O(e^4). \tag{39}$$

From (38) we can derive formulae for $\sin v$ and $\cos v$, namely

$$\sin v = \sin M + e\sin 2M - \tfrac{1}{8}e^2(7\sin M - 9\sin 3M) - \tfrac{1}{6}e^3(7\sin 2M - 8\sin 4M) + O(e^4) \tag{40}$$

and

$$\cos v = \cos M - e(1-\cos 2M) - \tfrac{9}{8}e^2(\cos M - \cos 3M) - \tfrac{4}{3}e^3(\cos 2M - \cos 4M) + O(e^4). \tag{41}$$

Finally, we generalize (40) and (41) to $O(e)$ expressions for the sine and cosine of $(jv+k\omega)$. Allowing T to denote either one of these functions, we have

$$T(jv+k\omega) = T(jM+k\omega) + je\,T\{(j+1)\,M+k\omega\} - je\,T\{(j-1)\,M+k\omega\} + O(e^2). \tag{42}$$

We can, of course, replace M by v, if desired, in the two je-terms. Also, by taking $j = k$, we can derive the relations between the sines and cosines of ku and kU.

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3. Mean elements and coordinates

3.1. *Mean elements*

Let ζ be an osculating element, i.e. it stands for any one of a, e, i, Ω, ω, σ, M, ξ, η and U. Its variation, due to the zonal harmonics, is made up of a long-term effect, not directly related to the orbital period (i.e. independent of u and v), and a series of short-periodic terms (i.e. of period less than the *orbital* period). The combination of the short-periodic terms is the short-periodic perturbation, $\delta\zeta$, removal of which from ζ gives a mean element, $\bar{\zeta}$, which for most purposes (because its variation can be plotted easily and accurately over long periods of time) is more useful than the osculating element; thus

$$\zeta = \bar{\zeta} + \delta\zeta. \tag{43}$$

It is a little more complicated with element variation due to the tesseral harmonics, since this variation depends on ν, the sidereal angle, as we shall see in (63). Thus the terms independent of M (equivalently u or v) are still of shortish period – they are known as m-daily terms – and are best included in the appropriate $\delta\zeta$. On the other hand, however, it is sometimes true that terms in which both M and ν appear vary only very slowly; this is the phenomenon of resonance, and it is allowed for by shifting the relevant effects from $\delta\zeta$ to $\bar{\zeta}$.

The variation of $\bar{\zeta}$ in general has two components, both due just to the zonal harmonics, namely a purely secular effect, which is zero for certain of the elements, and a long-periodic effect related to the secular variation of ω; in special cases, however, there will also be a component due to resonance. The essential difference between the long-periodic component of $\bar{\zeta}$, $\Delta\zeta$ say, and the short-periodic perturbations, $\delta\zeta$, follows at once from (43); thus if $\dot{\bar{\zeta}}$ denotes the secular rate of change of ζ,

$$\bar{\zeta} = \bar{\zeta}_0 + \dot{\bar{\zeta}}t + \Delta\zeta, \tag{44}$$

and so $\Delta\zeta$ is zero when $t = 0$; $\delta\zeta$ cannot in general be zero at epoch, however, unless special epochs (ascending nodes, for example) are chosen. This essential difference is often obscured by attempts to remove $\Delta\zeta$, as well as $\delta\zeta$, from ζ, the objection to this being that the term in $\Delta\zeta$ of longest period is then no longer bounded as this period tends to infinity. In particular, since $\dot{\bar{\omega}}$ vanishes at the so-called critical inclinations (Allan 1970) (given, as far as J_2 is concerned, by $f = 0.8$), an entirely specious singularity can be constructed in the various $\Delta\zeta$, of which more in § 9.1.

Another way to look at this is that, in the integration of the planetary equations, $\Delta\zeta$ must be obtained as a definite integral with lower limit $t = 0$, whereas $\delta\zeta$ can be expressed as an indefinite integral. As with every indefinite integral, an arbitrary constant is associated with each $\delta\zeta$, and this may be chosen as desired. As indicated in § 1.1, the choice may be unbiased, so as to make the value of $\delta\zeta$ zero when averaged with respect to t or v (with different results unless the orbit is circular), but a better idea is to choose the constants so as to simplify the expressions for a suitable set of derived quantities. This is the procedure adopted here, the 'suitable derived quantities' being the cylindrical polar coordinates that are introduced in § 3.3. The choice of constants is considered explicitly in regard to J_2-perturbations, first-order constants k_ζ being introduced in (138)–(146), and second-order constants $k_{2\zeta}$ in (209), (216) and (260). Section 4.2 provides a basis for inclusion of the effects due to other harmonics, in the first-order k_ζ.

3.2. *Satellite position*

The significance of a set of mean elements, $\bar{\zeta}$, is best understood from formulae for their direct use (in conjunction with perturbation formulae) to generate a satellite's position coordinates, namely x, y and z.

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The standard algorithm for x, y and z, given the *osculating* elements a, e, i, Ω, ω and M, is as follows:

(i) the eccentric anomaly, E, is found by solving Kepler's equation

$$E - e\sin E = M; \tag{45}$$

(ii) the true anomaly, v, is found from one of the two equivalent formulae (apart from an ambiguity of quadrant in the first formula)

$$\tan v = q\sin E/(\cos E - e) \tag{46}$$

and
$$\tan \tfrac{1}{2}v = [(1+e)/(1-e)]^{\frac{1}{2}} \tan \tfrac{1}{2}E; \tag{47}$$

(iii) u is obtained from (18), and r from either

$$r = p/(1+e\cos v) \tag{48}$$

or the equivalent formula
$$r = a(1-e\cos E); \tag{49}$$

(iv) x, y and z are obtained from the double coordinate transformation expressed by the matrix formula

$$(x \quad y \quad z)^{\mathrm{T}} = \mathrm{R}_3(-\Omega)\,\mathrm{R}_1(-i)\,(r\cos u \quad r\sin u \quad 0)^{\mathrm{T}}, \tag{50}$$

where $\mathrm{R}_j(\theta)$ describes rotation about the jth axis, so that

$$\mathrm{R}_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \text{ etc.}$$

(T, in (50), denotes transposition).

There is, of course, a preliminary step given by

$$M = M_0 + nt, \tag{51}$$

if the mean anomaly is only available at epoch (though appropriate for use at time t), and it may be necessary to recover e, ω and M from non-singular elements ξ, η and U defined by equations (1)–(3).

Now suppose the starting point is a set of *mean* elements at epoch, with long-term effects and short-periodic perturbations known. Then each ζ (osculating) can be obtained from (44) and (43) before operating the algorithm, but an alternative procedure is more satisfactory since it requires only four δ-expressions instead of six. In this procedure, as given by Kozai (1959), quantities δr and δu, representing short-periodic perturbations in r and u, appear instead of δa, δe, $\delta\omega$ and δM. Steps (i)–(iii) of the standard algorithm now operate on the *mean* elements, leading in turn to $\bar{E}$, $\bar{v}$, $\bar{u}$ and $\bar{r}$, after which, as an additional part of step (iii), r and u are derived as $\bar{r}+\delta r$ and $\bar{u}+\delta u$ respectively. The operation of step (iv) is exactly as before, δi and $\delta\Omega$ being incorporated in i and Ω as the third and fourth δ-expressions.

But Kozai's modification of the standard algorithm for x, y and z can be extended, to its logical conclusion, so that only *three* δ-expressions are required. Two of the perturbed quantities are slightly modified forms of r and u, and may thus be conveniently written r' and u', while the third represents the displacement of the satellite from its 'mean orbital plane', i.e. from a plane of inclination $\bar{\imath}$ and nodal right-ascension $\bar{\Omega}$. This displacement is in the 'cross-track direction' and may be conveniently denoted by c, though we cannot distinguish the perturbation δc from c itself, since the unperturbed value is zero. The positive direction for c is such that the (positive)

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direction of orbital motion is given by the right-hand screw rule, the first-order formula for δc being, in consequence,

$$\delta c = r(\delta i \sin \bar{u} - \delta \Omega \sin \bar{\imath} \cos \bar{u}). \tag{52}$$

This approach gives a complete first-order representation of δi via δc, but only represents the $\delta\Omega \sin \bar{\imath}$ component of $\delta\Omega$. The other component has to be incorporated with δu, the combined effect being $\delta u'$ and given to first order by

$$\delta u' = \delta u + \delta\Omega \cos \bar{\imath}. \tag{53}$$

The operation of step (iv) is different with this approach, since $\bar{\imath}$ and $\bar{\Omega}$, not i and Ω, must be used in the coordinate transformation. Thus (50) is replaced by the formula

$$(x \quad y \quad z)^{\mathrm{T}} = \mathrm{R}_3(-\bar{\Omega})\, \mathrm{R}_1(-\bar{\imath})\, (r' \cos u' \quad r' \sin u' \quad c)^{\mathrm{T}}. \tag{54}$$

Now

$$r'^2 + c^2 = r^2, \tag{55}$$

since they are equal to the same thing, namely $x^2+y^2+z^2$. It is from (55) that the first-order identity of r' and r follows. In the analysis of J_2-perturbations we find that, by appropriate choice of k_i and k_Ω, δc can be kept to $O(J_2 e, J_2^2)$; then $r'-r$ will be $O(J_2^2 e^2, J_2^3 e, J_2^4)$ and hence (for our purpose) entirely negligible.

3.3. *The system of cylindrical coordinates*

The quantities r', u' and c form a set of cylindrical polar coordinates relative to the mean orbital plane that is defined by $\bar{\imath}$ and $\bar{\Omega}$. (To a first approximation, $\bar{\imath}$ and $\dot{\bar{\Omega}}$ are constant, so that the plane has fixed inclination and rotates at a constant rate.) Relative to an instantaneous orientation of this plane, the directions of increasing r', increasing u' and increasing c are orthogonal. It is tempting, therefore, to replace the angular quantity, u', by a third linear quantity, l say, so as to have a (rotating) Cartesian triad. The connecting relation is

$$\delta l = r\,\delta u', \tag{56}$$

and the rotating Cartesian system works very well for first-order perturbations. It is unsatisfactory when second-order perturbations are required, however, the difficulty being that l (like ψ, L and ρ, but unlike u') is only defined differentially.

The first-order differential expressions (untruncated) for the cylindrical coordinates in terms of the elements are (bars on the elements are omitted)

$$\delta r' = (r/a)\,\delta a - a\delta e \cos v + (a/q)\, e\delta M \sin v, \tag{57}$$

$$\delta u' = \delta\omega + \delta\Omega \cos i + q^{-1}\delta M + (2/q^2)\,(\delta e \sin v + q^{-1} e\delta M \cos v) + (e/2q^2)\,[\delta e \sin 2v + q^{-1} e \delta M(3+\cos 2v)] \tag{58}$$

and

$$\delta c = r(\delta i \sin u - \delta\Omega \sin i \cos u). \tag{59}$$

If $O(e)$ terms can be ignored, the expressions simplify to

$$\delta r' = \delta a - a(\delta e \cos v + e\delta\psi \sin v), \tag{60}$$

$$\delta u' = 2(\delta e \sin v - e\delta\psi \cos v) + \delta L \tag{61}$$

and

$$\delta c = a(\delta i \sin u - \delta\Omega \sin i \cos u). \tag{62}$$

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4. First-order analysis with low eccentricity

4.1. *Variation of elements*

We require the effects of C_{nm} and S_{nm}; thus we have to integrate (27)–(32) with U given by the U_{nm} of (33) and with $O(e)$ effects ignored.

To express U_{nm} in terms of the desired arguments, we invoke the formula

$$P_l^m(\sin\beta)\exp(\mathrm{i}m\lambda) = \sum_{p=0}^{l} F_{lmp}(i)\exp\mathrm{i}\{(l-2p)\,u+m(\Omega-\nu)\}, \tag{63}$$

where $\mathrm{i} = (-1)^{\frac{1}{2}}$ and which effectively (Allan 1965) defines the inclination functions F; $F_{lmp}(i)$ is a real function when l and m have the same parity; otherwise it is pure imaginary. The F-functions may be generated by recurrence relations (Gooding 1971). The use of the index p in (63) has become fairly standard, but it is more symmetric, and certainly more useful, to work with an index k, such that (Gooding 1971)

$$k = l-2p. \tag{64}$$

Now from (34) and (35) we have

$$C_{nm}\cos m\lambda + S_{nm}\sin m\lambda = \mathrm{Re}\,\bar{J}_{nm}N_{lm}\exp\{\mathrm{i}m(\lambda-\lambda_{nm})\} \tag{65}$$

and hence, by using (63), (33) may be written as

$$U_{nm} = \frac{\mu}{r}\left(\frac{R}{r}\right)^n \bar{J}_{nm}\,\mathrm{Re}\sum_{p=0}^{l}[N_{lm}F_{lmp}(i)\exp\mathrm{i}\{(l-2p)u+m(\Omega-\nu-\lambda_{nm})\}]. \tag{66}$$

We introduce γ and χ, where

$$\gamma = \tfrac{1}{2}\pi - u \tag{67}$$

and

$$\chi = \tfrac{1}{2}\pi - \Omega + \nu + \lambda_{nm}, \tag{68}$$

since we can then write

$$\exp\mathrm{i}\{(l-2p)\,u+m(\Omega-\nu-\lambda_{nm})\} = \mathrm{i}^{k+m}\exp\mathrm{i}(-k\gamma-m\chi). \tag{69}$$

Finally, we define $\bar{F}_{lm}^{k}(i)$ by

$$\bar{F}_{lm}^{k}(i) = \mathrm{i}^{k+m}N_{lm}F_{lmp}(i), \tag{70}$$

where $p = \frac{1}{2}(l-k)$ by (64); this makes $\bar{F}_{lm}^{k}(i)$ always real, and we may write

$$U_{nm} = \frac{\mu}{r}\left(\frac{R}{r}\right)^n \bar{J}_{nm}\sum_{k=-l}^{l}[\bar{F}_{lm}^{k}(i)\cos(k\gamma+m\chi)]. \tag{71}$$

The summation on k is to be understood as only covering values with the same parity as l. When m = 0, equal and opposite values of k give the same contribution to U_{nm}, so only non-negative values need then be considered.

It is now natural to decompose U_{nm} into $\Sigma_k U_{nm}^k$, and convenient to write

$$U_{nm}^k = \frac{\mu}{r}\left(\frac{a}{r}\right)^n \hat{F}\cos(k\gamma+m\chi), \tag{72}$$

where

$$\hat{F} = \bar{J}_{nm}(R/a)^n\bar{F}_{lm}^{k}(i). \tag{73}$$

We shall also require $\hat{F}'$, defined as $\partial\hat{F}/\partial i$.

To obtain the appropriate derivatives for substituting in the planetary equations we need to observe that

$$\left(\frac{a}{r}\right)^{n+1} = 1+(n+1)\,e\cos v+O(e^2). \tag{74}$$

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Then $\dot{a} = 2na\mathrm{k}\hat{F}\sin(\mathrm{k}\gamma+\mathrm{m}\chi)$, (75)

$$\dot{e} = \tfrac{1}{2}n\hat{F}\{(\mathrm{n}+1+2\mathrm{k})\sin(\mathrm{k}\gamma+\mathrm{m}\chi-v)-(\mathrm{n}+1-2\mathrm{k})\sin(\mathrm{k}\gamma+\mathrm{m}\chi+v)\}, \quad (76)$$

$$\dot{i} = n\hat{F}\operatorname{cosec} i(\mathrm{k}\cos i-\mathrm{m})\sin(\mathrm{k}\gamma+\mathrm{m}\chi), \quad (77)$$

$$\dot{\Omega} = n\hat{F}'\operatorname{cosec} i\cos(\mathrm{k}\gamma+\mathrm{m}\chi), \quad (78)$$

$$e\dot{\psi} = \tfrac{1}{2}n\hat{F}\{(\mathrm{n}+1+2\mathrm{k})\cos(\mathrm{k}\gamma+\mathrm{m}\chi-v)+(\mathrm{n}+1-2\mathrm{k})\cos(\mathrm{k}\gamma+\mathrm{m}\chi+v)\} \quad (79)$$

and $\dot{\rho} = 2n\hat{F}(\mathrm{n}+1)\cos(\mathrm{k}\gamma+\mathrm{m}\chi)$. (80)

We now obtain perturbations by integrating the above equations on the basis that the right-hand sides are constant apart from linear variations in γ, χ and v. Furthermore, $\dot{\bar{\gamma}} = -(\bar{n}+\dot{\bar{\omega}})$, and we define

$$\mathrm{k}' = \mathrm{k}(1+\dot{\bar{\omega}}/\bar{n}) \quad (\approx \mathrm{k}), \quad (81)$$

so that $\mathrm{k}\dot{\bar{\gamma}} = -\mathrm{k}'\bar{n}$. Also, $\dot{\bar{\chi}}$ is $\dot{\nu}-\dot{\bar{\Omega}}$, and we define

$$s = (\dot{\nu}-\dot{\bar{\Omega}})/(\bar{n}+\dot{\bar{\omega}}) \quad (\approx \dot{\nu}/\bar{n}) \quad (82)$$

and

$$\mathrm{m}' = \mathrm{m}(1+\dot{\bar{\omega}}/\bar{n}) \quad (\approx \mathrm{m}) \quad (83)$$

so that $\mathrm{m}\dot{\bar{\chi}} = \mathrm{m}'s\bar{n}$; clearly s is the ratio of the Earth's rate of rotation (relative to the satellite's orbital plane) to the satellite's rate of (nodal) revolution. The perturbations may be concisely expressed as indefinite integrals, though (as indicated in §§ 1.2 and 3.1) the implied choice of arbitrary constants is not always the best. Then

$$\delta a = \frac{2a\mathrm{k}\hat{F}}{\mathrm{k}'-\mathrm{m}'s}\cos(\mathrm{k}\gamma+\mathrm{m}\chi), \quad (84)$$

$$\delta e = \tfrac{1}{2}\hat{F}\left\{\frac{\mathrm{n}+1+2\mathrm{k}}{\mathrm{k}'-\mathrm{m}'s+1}\cos(\mathrm{k}\gamma+\mathrm{m}\chi-v)-\frac{\mathrm{n}+1-2\mathrm{k}}{\mathrm{k}'-\mathrm{m}'s-1}\cos(\mathrm{k}\gamma+\mathrm{m}\chi+v)\right\}, \quad (85)$$

$$\delta i = \frac{\hat{F}\operatorname{cosec} i(\mathrm{k}\cos i-\mathrm{m})}{\mathrm{k}'-\mathrm{m}'s}\cos(\mathrm{k}\gamma+\mathrm{m}\chi), \quad (86)$$

$$\delta\Omega = -\frac{\hat{F}'\operatorname{cosec} i}{\mathrm{k}'-\mathrm{m}'s}\sin(\mathrm{k}\gamma+\mathrm{m}\chi), \quad (87)$$

$$e\delta\psi = -\tfrac{1}{2}\hat{F}\left\{\frac{\mathrm{n}+1+2\mathrm{k}}{\mathrm{k}'-\mathrm{m}'s+1}\sin(\mathrm{k}\gamma+\mathrm{m}\chi-v)+\frac{\mathrm{n}+1-2\mathrm{k}}{\mathrm{k}'-\mathrm{m}'s-1}\sin(\mathrm{k}\gamma+\mathrm{m}\chi+v)\right\} \quad (88)$$

and

$$\delta\rho = -\frac{2\hat{F}(\mathrm{n}+1)}{\mathrm{k}'-\mathrm{m}'s}\sin(\mathrm{k}\gamma+\mathrm{m}\chi). \quad (89)$$

Formulae for perturbations in the other elements referred to in § 2.1, namely ω, σ, M, ξ, η, U and L, follow from the formulae above, the only difficulty being in the dependence of δM, δU and δL on $\int\delta n\,\mathrm{d}t$. Since $\delta n = -1\tfrac{1}{2}(n/a)\,\delta a$, however, we have at once that

$$\int\delta n\,\mathrm{d}t = \frac{3\mathrm{k}\hat{F}}{(\mathrm{k}'-\mathrm{m}'s)^2}\sin(\mathrm{k}\gamma+\mathrm{m}\chi). \quad (90)$$

For the perturbation in L, given by $\delta L = \delta\rho+\int\delta n\,\mathrm{d}t$, we then have

$$\delta L = \frac{\hat{F}}{(\mathrm{k}'-\mathrm{m}'s)^2}\{3\mathrm{k}-2(\mathrm{n}+1)(\mathrm{k}'-\mathrm{m}'s)\}\sin(\mathrm{k}\gamma+\mathrm{m}\chi). \quad (91)$$

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The derivation of (84)–(89) is valid for general values of m′, k′ and s, the perturbations these formulae express being short-periodic or m-daily, and the integration constants zero. Special cases arise when a denominator is zero, or 'close to zero' according to some criterion. Then bounded perturbations can only be ensured by returning to (75)–(80) and, as remarked in § 3.1, by evaluating definite integrals. The resulting perturbations are 'long term' and so contribute to $\Delta\zeta$, rather than $\delta\zeta$, in the notation of § 3.1. Clearly, the special cases for e and ψ are different from those for a, i, Ω and ρ.

The most obvious special case is for k = m = 0. Then (75) and (77) give zero, integrating to constant terms in δa and δi respectively, and we shall see in § 4.2 that we do not want these constants to be zero. Also (78) and (80) give 'pure secular' terms in Ω and ρ respectively, i.e. contribute to $\dot{\bar{\Omega}}$ and $\dot{\bar{\rho}}$; since we are working with $e\psi$ and $\dot{\rho}$, we do not directly encounter the familiar secular perturbation in ω that underlies the use of $\dot{\bar{\omega}}$ in (81)–(83), the basic effect of $\dot{\bar{\omega}}$ being covered by $\dot{\bar{\rho}}$, as (5)–(7) indicate. The term in $\dot{\bar{\rho}}$ is $2n\hat{F}(\mathrm{n}+1)$ and this may be conventionally included in $\bar{n}$, at the expense of losing the Kepler relation (4) between $\bar{a}$ and $\bar{n}$.

When k = ± 1, m still being zero, special cases arise for one of the terms in each of (76) and (79), these terms being proportional to $\cos\omega$ and $\sin\omega$ respectively. The terms for k = 1 are just the 'opposite pair' of those for k = − 1, this being an illustration of the remark that, when m = 0, only non-negative values of k need be retained. Integration gives long-periodic perturbations, confined to terms in $\sin\omega$ and $\cos\omega$ as a result of our neglect of perturbations that are $O(J_{\mathrm{nm}}e)$.

When m ≠ 0, we can only get small denominators for special values of s. This is the classical condition of resonance, and it occurs (for the appropriate elements) whenever $\mathrm{k}-\mathrm{m}s$ is close to 0, 1 or − 1 (we must remember that only values of k having the same parity as l are legitimate). If $\mathrm{k}-\mathrm{m}s \approx 0$, the effect of resonance is a quasi-secular variation of semi-major axis (a), with a consequent quasi-quadratic along-track variation (i.e. in L). If $\mathrm{k}-\mathrm{m}s \approx \pm 1$, there is quasi-secular variation of e and ω.

4.2. *Perturbations in cylindrical coordinates*

As explained in § 3.3, we can derive first-order perturbations in the set of cylindrical polar coordinates. Substituting (84)–(88) and (91) in (60)–(62), we get, for the effect of the $\mathrm{U}^{\mathrm{k}}_{\mathrm{nm}}$ potential with $O(e)$ terms neglected,

$$\delta r' = \frac{a\hat{F}\{2\mathrm{k}-(\mathrm{n}+1)(\mathrm{k}'-\mathrm{m}'s)\}}{(\mathrm{k}'-\mathrm{m}'s)\{1-(\mathrm{k}'-\mathrm{m}'s)^2\}}\cos(\mathrm{k}\gamma+\mathrm{m}\chi), \tag{92}$$

$$\delta u' = \frac{\hat{F}\{3\mathrm{k}-2(\mathrm{n}+1)(\mathrm{k}'-\mathrm{m}'s)+\mathrm{k}(\mathrm{k}'-\mathrm{m}'s)^2\}}{(\mathrm{k}'-\mathrm{m}'s)^2\{1-(\mathrm{k}'-\mathrm{m}'s)^2\}}\sin(\mathrm{k}\gamma+\mathrm{m}\chi) \tag{93}$$

and

$$\delta c = \frac{a\operatorname{cosec} i}{2(\mathrm{k}'-\mathrm{m}'s)}\{[\hat{F}'\sin i+\hat{F}(\mathrm{k}\cos i-\mathrm{m})]\cos(\mathrm{k}\gamma-\gamma+\mathrm{m}\chi) - [\hat{F}'\sin i-\hat{F}(\mathrm{k}\cos i-\mathrm{m})]\cos(\mathrm{k}\gamma+\gamma+\mathrm{m}\chi)\}. \tag{94}$$

The formulae obviously break down when they involve zero denominators, arising from the zero denominators in δa, δe, etc. that were considered in § 4.1. Valid formulae for the various types of special case can always be obtained by returning to (75)–(80), before invoking (60)–(62).

For k = m = 0, the δe and $\delta\psi$ terms of (60) contribute $-a\hat{F}(\mathrm{n}+1)$ to $\delta r'$, i.e. a constant. This can be eliminated by off-setting the 'mean' semi-major axis, $\bar{a}$, by the same amount, effectively adopting $\delta a = a\hat{F}(\mathrm{n}+1)$ as the integral of $\dot{a} = 0$; for n = 2, this gives the e-independent part of the 'preferred' value of the constant k_a which we meet in § 6 – see equations (138) and (160). Since $\delta\rho$ goes (via δL in (61)) directly into $\delta u'$, we can eliminate a secular term in u' by an off-set of

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$2n\hat{F}(\mathrm{n}+1)$ in the 'mean mean motion', $\bar{n}$, as already suggested in § 4.1. The effect of these two offsets is that we require

$$\bar{n}^2\bar{a}^3 = \mu\{1+\hat{F}(\mathrm{n}+1)\} \tag{95}$$

as the expression of Kepler's third law. This is valid for first-order J_n-perturbations, but to go beyond the first order, we should work with U rather than the unreal L. Thus if $\bar{n}$ is to stand for $\dot{\bar{U}}$, it is less by $\dot{\bar{\Omega}}\cos i$ than the $\bar{n}$ of (95); the expression for Kepler's third law is then, on using (78),

$$\bar{n}^2\bar{a}^3 = \mu\{1+\hat{F}(\mathrm{n}+1)-2\hat{F}'\cot i\}. \tag{96}$$

For n = 2, this gives the formula we shall meet as equation (190).

For k = 2, with m = 0 (not a zero-denominator case, but of interest notwithstanding), the first term of (94) gives $\frac{1}{4}a(\hat{F}'+2\hat{F}\cot i)\sin u$, and the same quantity comes from the second term of (94) for k = −2. But a constant component of i contributes a term in $\sin u$ to δc, according to (62), so it is only necessary to take this constant (in i) as $-\frac{1}{2}(\hat{F}'+2\hat{F}\cot i)$, on evaluating for k = 2, to eliminate the term in $\sin u$ (in c); for n = 2, this gives the 'preferred' value of the constant k_i which we meet in § 6 – see equations (140) and (170).

5. Perturbations associated with the Sun and Moon

As remarked in § 2.3, perturbations due to solar radiation (with eclipse absent) and lunisolar gravity can be derived by appropriate interpretation of J_n-coefficients with negative n. It is necessary to assume a fixed position for the external body in question, its coordinates being specified by a distance R (the symbol 'R' has a new meaning now, having previously denoted Earth radius) and direction cosines (A, B, C) – an 'orbital system' of axes is assumed here, with first axis towards the satellite's ascending node and third axis normal to the orbital plane, as used by Gooding (1966) and Cook (1962).

A complication arises because, whichever external body is involved, the J_n have to be interpreted as zonal harmonics relative to a plane which is not the Earth's equator. The appropriate plane may be thought of as a quasi-equator, however; it is the plane perpendicular to the direction of the given external body, such that this direction is then quasi-north. Thus the general formulae for perturbations δa, etc. give expressions for $\delta i'$ and $\delta\Omega'$, rather than δi and $\delta\Omega$, but the transformation to δi and $\delta\Omega$ is straightforward (Gooding 1966).

There is a simpler approach to the derivation of the formulae for $\delta r'$, $\delta u'$ and δc, however. We derive these initially as if the quasi-equator were the genuine equator, i.e. as if the disturbing body were in the north-polar direction, and we observe that in the rotating Cartesian axes of § 3.3 the direction cosines of the polar direction are $(\sin i\sin u, \sin i\cos u, \cos i)$, whereas the direction cosines of the disturbing body's direction are $(A\cos u+B\sin u, -A\sin u+B\cos u, C)$. (These axes may be described as 'satellite axes', being rotated through angle u, about the normal to the orbital plane, from 'orbital axes'.) We introduce the pair of polar angles, θ and ϕ, respectively 'latitude' and 'longitude' relative to orbital axes, such that

$$(A, B, C) = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta). \tag{97}$$

(In terms of the α and i' of § 8.3 of Gooding (1966), $\phi = \alpha+\frac{1}{2}\pi$ and $\theta = \frac{1}{2}\pi-i'$.) The direction cosines of the disturbing body, i.e. of the quasi-polar direction, are now $(\cos\theta\cos(u-\phi)$, $-\cos\theta\sin(u-\phi)$, $\sin\theta)$ relative to 'satellite axes'. Hence the formulae for $\delta r'$, $\delta u'$ and δc can be

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obtained by simple replacement of $\sin i \sin u$, $\sin i \cos u$ and $\cos i$ by $\cos\theta \cos(u-\phi)$, $-\cos\theta \times \sin(u-\phi)$ and $\sin\theta$, respectively; since ω is the value of u corresponding to perigee, the equivalent replacement must be made for formulae involving ω.

5.1. *Application of J_{-2} to solar radiation*

In the absence of eclipse, the effects of solar radiation may be regarded as arising from a constant force, with a potential corresponding to a J_{-2} given by

$$J_{-2} = S/\mu, \tag{98}$$

where S is defined such that S/R^2 is the force per unit mass acting on the satellite. The value of S will of course vary greatly from one satellite to another.

When $\mathrm{n} = -2$, then $\mathrm{l} = 1$, so only two values of k arise, namely 1 and -1; in fact we only need consider $\mathrm{k} = 1$, since the coefficients in the formulae can be doubled. We have (for $\mathrm{k} = 1$), taking $N_{-2} = -1$,

$$\hat{F} = -\tfrac{1}{2}J_{-2}(a/R)^2 \sin i, \tag{99}$$

where R is the distance from the Sun.

5.2. *Application of J_{-3} to lunisolar gravity*

The gravitational effect of a disturbing body being 'differential', we require n to be -3, rather than -2, and take, as in Gooding (1966),

$$J_{-3} = -\mu_{\mathrm{d}}/\mu, \tag{100}$$

where μ_{d} is the product of the disturbing body's mass and the constant of gravitation. Now $\mathrm{n} = -3$ means that $\mathrm{l} = 2$, so we have

$$\hat{F} = \tfrac{1}{2}J_{-3}(a/R)^3 h \quad \text{for} \quad \mathrm{k} = 0 \tag{101}$$

and

$$\hat{F} = -\tfrac{3}{8}J_{-3}(a/R)^3 f \quad \text{for} \quad \mathrm{k} = 2. \tag{102}$$

5.3. *Application of J_{-4} to parallactic term of lunar gravity*

Whereas the ratio of lunar effects to solar effects is about 2.0 for the principal terms of the respective potentials, for the next term – the parallactic term – the ratio exceeds 750. Thus we only need be concerned with the Moon, when considering this (and any subsequent) term. We have (with $J_{-4} = J_{-3}$)

$$\hat{F} = \tfrac{3}{16}J_{-4}(a/R)^4 (4-5f) \sin i \quad \text{for} \quad \mathrm{k} = 1 \tag{103}$$

and

$$\hat{F} = -\tfrac{5}{16}J_{-4}(a/R)^4 f \sin i \quad \text{for} \quad \mathrm{k} = 3. \tag{104}$$

5.4. *Remark*

Neglect of the Moon's motion is likely to be as serious as would be neglect of the parallactic term. However, it is possible to allow for the effect of lunar motion quite easily, so far as the principal term in the potential expansion is concerned, if it is assumed that this motion is uniform. The underlying idea is that the effect of a J_{-3} field with moving axis is equivalent to the combination of a J_{-3} field of fixed axis, namely the axis of the (revolutionary) motion, and a (rotating) $J_{-3,2}$ field. To apply this idea to the motion of the Moon, we take the direction normal to the lunar orbit as our fixed axis, with $J_{-3} = \mu_{\mathrm{M}}/2\mu$ and $J_{-3,2} = \mu_{\mathrm{M}}/4\mu$, $\lambda_{-3,2}$ being zero.

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Perturbations due to this J_{-3} are dealt with as in § 5.2, with θ and ϕ re-interpreted and numerical coefficients multiplied by -0.5, but it is more tricky to cope with $J_{-3,2}$. In principle $J_{-3,2}$ behaves like $J_{2,2}$, but we have a field rotation rate of 13.18°/day (the lunar motion) instead of 361°/day.

6. First-order analysis with unrestricted eccentricity

6.1. *Effect of the general zonal harmonic*

In this section we no longer regard $O(e)$ perturbations as negligible, but we stop the analysis from becoming too complex by restricting it to zonal harmonics; however, the suffix n (of J_n) is still allowed to be negative. We do the analysis by replacing t, as independent variable in Lagrange's planetary equations, by v. In this method v has to be interpreted as 'true anomaly in an unperturbed orbit', since the relation between v and t that we require, to transform the independent variable, is

$$\dot{v} = nq(a/r)^2, \tag{105}$$

and this is only exact for an unperturbed orbit. The distinction need not concern us, however, since the analysis is not yet proceeding beyond first order in the J-coefficients.

The planetary equations can now be written as follows:

$$\frac{da}{dv} = \frac{2r^2}{\mu q}\frac{\partial U}{\partial M}, \tag{106}$$

$$\frac{de}{dv} = \frac{r^2}{\mu ae}\left(q\frac{\partial U}{\partial M} - \frac{\partial U}{\partial \omega}\right), \tag{107}$$

$$\frac{di}{dv} = \frac{r^2 \operatorname{cosec} i}{\mu aq}\left(\cos i\frac{\partial U}{\partial \omega} - \frac{\partial U}{\partial \Omega}\right), \tag{108}$$

$$\frac{d\Omega}{dv} = \frac{r^2 \operatorname{cosec} i}{\mu aq}\frac{\partial U}{\partial i}, \tag{109}$$

$$e\frac{d\psi}{dv} = \frac{r^2}{\mu a}\frac{\partial U}{\partial e} \tag{110}$$

and

$$\frac{d\rho}{dv} = -\frac{2r^2}{\mu q}\frac{\partial U}{\partial a}. \tag{111}$$

The expression we require for U is given by equation (20) of Gooding (1966) and is conveniently denoted by U_n^k. It is essentially U_{nm}^k of § 4.1, with m = 0, except that we can dispense with negative values of k by doubling U_{n0}^k when k > 0. We may in fact write

$$U_n^k = -\frac{\mu}{r}\left(\frac{p}{r}\right)^n \hat{A} \sin^k i \cos k\gamma, \tag{112}$$

where

$$\hat{A} \sin^k i = -(a/p)^n u_k \hat{F}, \tag{113}$$

u_k being 1 if k = 0 and 2 if k ≠ 0. In terms of the 'normalized' A-functions of Gooding (1966), we have

$$\hat{A} = J_n\left(\frac{R}{p}\right)^n \frac{u_k P_l^k(0)}{2^k k!} A_l^k(i). \tag{114}$$

We also require $\hat{D}$, similarly related to the D-functions (Gooding 1966), and effectively defined by

$$\hat{D} \sin^{k-1} i = -\left(\frac{a}{p}\right)^n u_k \frac{\partial \hat{F}}{\partial i}. \tag{115}$$

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To evaluate the partial derivatives of U_n^k with respect to a, e, i, Ω, ω and M, for insertion in the planetary equations, U_n^k being known as a function of i, r and γ (not p, since $p^n\hat{A}$ is free of p), we require the following results for a Kepler orbit:

$$\frac{\partial r}{\partial a} = \frac{r}{a}, \quad \frac{\partial r}{\partial e} = -a\cos v, \quad \frac{\partial r}{\partial M} = \frac{ae\sin v}{q},$$

$$\frac{\partial \gamma}{\partial e} = -\frac{\sin v(2+e\cos v)}{q^2}, \quad \frac{\partial \gamma}{\partial \omega} = -1 \quad \text{and} \quad \frac{\partial \gamma}{\partial M} = -\frac{a^2 q}{r^2}.$$

Evaluation of the derivatives being straightforward, though tedious, we eventually express the planetary equations as follows:

$$\frac{\mathrm{d}a}{\mathrm{d}v} = -\frac{a^2}{p}\hat{A}\sin^k i\left(\frac{p}{r}\right)^n\{2k\sin k\gamma + e[(n+1+k)\sin(k\gamma-v)-(n+1-k)\sin(k\gamma+v)]\}, \tag{116}$$

$$\frac{\mathrm{d}e}{\mathrm{d}v} = -\tfrac{1}{2}\hat{A}\sin^k i\left(\frac{p}{r}\right)^{n-1}\left\{\left[(n+1+k)\frac{p}{r}+k\right]\sin(k\gamma-v) - \left[(n+1-k)\frac{p}{r}-k\right]\sin(k\gamma+v)+2ek\sin k\gamma\right\}, \tag{117}$$

$$\frac{\mathrm{d}i}{\mathrm{d}v} = -k\hat{A}\sin^{k-1} i\cos i\left(\frac{p}{r}\right)^{n-1}\sin k\gamma, \tag{118}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}v} = -\hat{D}\sin^{k-2} i\left(\frac{p}{r}\right)^{n-1}\cos k\gamma, \tag{119}$$

$$e\frac{\mathrm{d}\psi}{\mathrm{d}v} = -\tfrac{1}{2}\hat{A}\sin^k i\left(\frac{p}{r}\right)^{n-1}\left\{\left[(n+1+k)\frac{p}{r}+k\right]\cos(k\gamma-v)+\left[(n+1-k)\frac{p}{r}-k\right]\cos(k\gamma+v)\right\} \tag{120}$$

and

$$\frac{\mathrm{d}\rho}{\mathrm{d}v} = -2(n+1)\,q\hat{A}\sin^k i\left(\frac{p}{r}\right)^{n-1}\cos k\gamma. \tag{121}$$

For completeness we need the expression for $\mathrm{d}L/\mathrm{d}v$. It is given by

$$\frac{\mathrm{d}L}{\mathrm{d}v} = \frac{\mathrm{d}\rho}{\mathrm{d}v} + \frac{n}{\dot{v}}, \tag{122}$$

where the formula for $\dot{v}$ was given at the beginning of this section.

The right-hand side of each of the above equations can be expressed in terms of v, since $\gamma = \frac{1}{2}\pi-\omega-v$ and $p/r = 1+e\cos v$, so that (at least in principle) we can obtain the perturbations δa, etc. by quadrature. When $n > 0$, there is no need to truncate at any power of e, since the right-hand sides of the equations terminate as power series in e, i.e. they are polynomials. For $n < 0$ this is not so. To obtain integrable functions of v, repeated appeal is necessary to fundamental trigonometrical identities such as

$$2\cos A\cos B = \cos(A+B)+\cos(A-B).$$

The general analysis will be taken no further, our real interest lying with J_2.

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6.2. *Perturbations in the elements due to J_2 alone*

We require to integrate (116)–(121) for U_2^0 and U_2^2 given by (112). For k = 0, we have, for $\hat{A}$ and $\hat{D}$ defined by (114) and (115),

$$\hat{A} = -\tfrac{1}{3}Kh \quad \text{and} \quad \hat{D} = Kf\cos i, \tag{123}$$

where f and h are given by (14) and (15), and K by

$$K = \tfrac{3}{2}J_2(R/p)^2; \tag{124}$$

K, with this meaning, will be used throughout the rest of the paper. For k = 2, similarly, we have

$$\hat{A} = \tfrac{1}{2}K \quad \text{and} \quad \hat{D} = K\cos i. \tag{125}$$

The first-order perturbations due to J_2 have been given by a number of authors, for example, by Roy (1978), Brouwer & Clemence (1961) and Sterne (1960), but are repeated here for completeness – there is some redundancy in that expressions are given for quantities that are not independent. Also, we introduce the constants k_ζ, referred to in §§ 1.2 and 3.1.

As there are no long-periodic effects of first order, the variation of the mean elements is purely secular, and the rates of change are as follows:

$$\dot{\bar{\Omega}} = -Kn\cos i, \tag{126}$$

$$\dot{\bar{\psi}} = Knh, \tag{127}$$

$$\dot{\bar{\rho}} = 2Knqh, \tag{128}$$

$$\dot{\bar{\omega}} = 2Kn(1-\tfrac{5}{4}f), \tag{129}$$

$$\dot{\bar{\sigma}} = Knqh, \tag{130}$$

$$\dot{\bar{M}} = n' \tag{131}$$

and

$$\dot{\bar{U}} = n' + \dot{\bar{\omega}}. \tag{132}$$

The 'mean mean motion', n', that appears in (131) and (132) is an absolute constant of the motion that is related to the energy intergral, as explained by Merson (1961) and Gooding (1966) (and used here in § 7.1). It is given by

$$n'^2 a'^3 = \mu, \tag{133}$$

where the relation between a' and (osculating) a is given by

$$a'^{-1} = a^{-1}(1 + 2a\mathrm{U}/\mu), \tag{134}$$

for

$$\mathrm{U} = \tfrac{1}{4}\mu J_2(R^2/r^3)\{2 - 3f(1-\cos 2u)\}. \tag{135}$$

Thus

$$a = a' + Ka'q^{-2}(p/r)^3(f\cos 2u + \tfrac{2}{3}h), \tag{136}$$

this being an exact result. Warning: (131) is fortuitous in that for n = 2 two terms happen to cancel; for general J_n, we have an additional term in $\bar{M}$, namely

$$-q\dot{\bar{\psi}} + [(2\mathrm{n}-1)/2(\mathrm{n}+1)]\,\dot{\bar{\rho}}. \tag{137}$$

It is also worth remarking that quantities on the right-hand sides of (126)–(130), as with (138)–(146) following, should be regarded as *mean*, so that $\bar{K}$, $\bar{\imath}$, etc. should really be written throughout; being only first-order equations, however, they are actually correct as they stand.

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In the formulae for the short-periodic perturbations, δa, etc., compactness is achieved by writing C_j for $\cos(jv+2\omega)$ and S_j for $\sin(jv+2\omega)$. Also v_M is introduced as an abbreviation for $v-M$; terms in v_M arise because the secular rates given by (126)–(130) really arise as $d\bar{\zeta}/d\bar{v}$ rather than $d\bar{\zeta}/dt$. The formulae are:

$$\begin{aligned}\delta a = \tfrac{1}{24}Kaq^{-2}\{&24(k_a+fC_2)+e[48h\cos v+36f(C_1+C_3)]\\&+e^2[24h\cos 2v+18f(2C_2+C_4)]\\&+e^3[4h(3\cos v+\cos 3v)+3f(C_{-1}+3C_1+3C_3+C_5)]\},\end{aligned} \tag{138}$$

$$\begin{aligned}\delta e = \tfrac{1}{48}K\{&48h\cos v+4f(3C_1+7C_3)+6e(4h\cos 2v+10fC_2+3fC_4+k_e)\\&+e^2[4h(3\cos v+\cos 3v)+f(3C_{-1}+33C_1+17C_3+3C_5)]\},\end{aligned} \tag{139}$$

$$\delta i = \tfrac{1}{12}K\sin 2i\{3(C_2+k_i)+e(3C_1+C_3)\}, \tag{140}$$

$$\delta\Omega = \tfrac{1}{6}K\cos i\{-6v_M+3(S_2+k_\Omega)-e(6\sin v-3S_1-S_3)\}, \tag{141}$$

$$\begin{aligned}\delta\psi = \tfrac{1}{48}Ke^{-1}\{&48h\sin v-4f(3S_1-7S_3)\\&+6e[4h(2v_M+\sin 2v)+3f(2S_2+S_4)+k_\psi]\\&+e^2[4h(9\sin v+\sin 3v)-f(3S_{-1}-21S_1-11S_3-3S_5)]\},\end{aligned} \tag{142}$$

$$\delta\rho = \tfrac{1}{2}Kq\{4hv_M+3fS_2+\tfrac{1}{2}k_\rho+e(4h\sin v+3fS_1+fS_3)\}, \tag{143}$$

$$\begin{aligned}\delta\omega = \tfrac{1}{48}Ke^{-1}\{&48h\sin v-4f(3S_1-7S_3)\\&+6e[4(4-5f)v_M+4h\sin 2v-2(2-5f)S_2+3fS_4+k_\omega]\\&+e^2[6(14-17f)\sin v+4h\sin 3v-3fS_{-1}-3(8-15f)S_1-(8-19f)S_3+3fS_5]\},\end{aligned} \tag{144}$$

$$\begin{aligned}\delta\sigma = -\tfrac{1}{48}Ke^{-1}q\{&48h\sin v-4f(3S_1-7S_3)\\&+6e[-8hv_M+4h\sin 2v-6fS_2+3fS_4-k_\sigma]\\&+e^2[-60h\sin v+4h\sin 3v-f(3S_{-1}+51S_1+13S_3-3S_5)]\}\end{aligned} \tag{145}$$

and

$$\begin{aligned}\delta M = -\tfrac{1}{48}Ke^{-1}q\{&48h\sin v-4f(3S_1-7S_3)+6e(4h\sin 2v+3fS_4+k_M)\\&-e^2[4h(3\sin v-\sin 3v)+f(3S_{-1}+15S_1+S_3-3S_5)]\}.\end{aligned} \tag{146}$$

The presence of the q-factor in (145) and (146) makes a simple complete expression for δU impossible; complete expressions for $\delta\xi$ and $\delta\eta$ are simple enough, however, but will not be given in this paper.

We now truncate the short-periodic perturbations to $O(e)$, to obtain expressions that we verify and use in § 7. For each element ζ we express $\delta\zeta$ by writing

$$\delta\zeta = \bar{K}\zeta_1 \tag{147}$$

and we drop e_1, ψ_1, ρ_1, ω_1, σ_1 and M_1 in favour of ξ_1, η_1, U_1 and n_1 – the reason for having n_1 as well as a_1 will become apparent. The expressions are:

$$a_1 = \tfrac{1}{2}\bar{a}\{2(k_a+\bar{f}\cos 2\bar{u})+\bar{e}[4\bar{h}\cos\bar{v}+3\bar{f}\cos(\bar{u}+\bar{\omega})+3\bar{f}\cos(2\bar{u}+\bar{v})]\}, \tag{148}$$

$$i_1 = \tfrac{1}{12}\sin 2\bar{i}\{3(\cos 2\bar{u}+k_i)+\bar{e}[3\cos(\bar{u}+\bar{\omega})+\cos(2\bar{u}+\bar{v})]\}, \tag{149}$$

$$\Omega_1 = \tfrac{1}{6}\cos\bar{i}\{3(\sin 2\bar{u}+k_\Omega)-\bar{e}[18\sin\bar{v}-3\sin(\bar{u}+\bar{\omega})-\sin(2\bar{u}+\bar{v})]\}, \tag{150}$$

$$\begin{aligned}\xi_1 = \tfrac{1}{24}\{&2[3(4\bar{h}+\bar{f})\cos\bar{u}+7\bar{f}\cos 3\bar{u}]\\&+3\bar{e}[6(1-\bar{f})\cos(\bar{u}+\bar{v})-2(1-5\bar{f})\cos(2\bar{u}+\bar{\omega})+3\bar{f}\cos(3\bar{u}+\bar{v})+k_\xi]\},\end{aligned} \tag{151}$$

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$$\eta_1 = \tfrac{1}{24}\{2[3(4\bar{h}-\bar{f})\sin\bar{u}+7\bar{f}\sin 3\bar{u}] + 3\bar{e}[2(1-3\bar{f})\sin(\bar{u}+\bar{v})-2(1-5\bar{f})\sin(2\bar{u}+\bar{\varpi})+3\bar{f}\sin(3\bar{u}+\bar{v})+k_\eta]\}, \tag{152}$$

$$U_1 = -\tfrac{1}{24}\{6(2-5\bar{f})\sin 2\bar{u}-\bar{e}[6(26-33\bar{f})\sin\bar{v} - 3(4-9\bar{f})\sin(\bar{u}+\bar{\varpi})-(4-17\bar{f})\sin(2\bar{u}+\bar{v})]\} \tag{153}$$

and
$$n_1 = -\tfrac{3}{4}\bar{n}\{2(k_n+\bar{f}\cos 2\bar{u})+\bar{e}[4\bar{h}\cos\bar{v}+3\bar{f}\cos(\bar{u}+\bar{\varpi})+3\bar{f}\cos(2\bar{u}+\bar{v})]\}. \tag{154}$$

(Important: in the untruncated expression for δn, corresponding to (138) for δa, the factor q^{-2} appears and, in particular, acts on k_n.) The coefficient of $\sin\bar{v}$ in (150) is three times as large as in (141), it will be noted. The difference is due to the e-term in (39), which comes in when v_M in (141) is expanded.

The quantity $\bar{n}$ that appears in (154) is a mean element that, like all the mean elements, is arbitrary to the extent of a constant, namely the constant k_n. In view of (131), it might seem natural just to take $\bar{n} = n'$, since then $\bar{n} = \dot{M}$, but alternative assumptions may be preferable, and will be considered in the next section.

Since (151) and (152) are equivalent to truncated versions of (139) and (144), we must have

$$k_e = k_\xi \cos\bar{\varpi}+k_\eta\sin\bar{\varpi} \tag{155}$$

and
$$k_\omega = -k_\xi\sin\bar{\varpi}+k_\eta\cos\bar{\varpi}. \tag{156}$$

(Also k_ψ, k_ρ and k_σ are equal to $k_\omega+4(1-\bar{f})\,k_\Omega$, $k_\psi-k_M$ and $k_\psi-2k_M$, respectively, but k_ψ, k_ρ and k_σ will not be used again.) It is because of the e^{-1}-factor in (144) that the non-singular ξ_1 and η_1 should be used instead of e_1 and ω_1, but $\dot{\bar{\varpi}}$ in (129) is free of this factor – in other words, $\dot{\bar{\varpi}}$ only has an $O(Ke)$ effect, apart from its action in combination with n' in (132). It follows that the most satisfactory way to generate (osculating) e and ω from (mean at epoch) $\bar{e}_0$ and $\bar{\varpi}_0$ is by combining formulae for $\dot{\bar{\varpi}}$, ξ_1 and η_1, such that

$$\xi = \bar{e}_0\cos(\bar{\varpi}_0+\dot{\bar{\varpi}}t)+\bar{K}\xi_1 \tag{157}$$

and
$$\eta = \bar{e}_0\sin(\bar{\varpi}_0+\dot{\bar{\varpi}}t)+\bar{K}\eta_1, \tag{158}$$

from which e and ω follow by inversion of (1) and (2).

6.3. *Perturbations in coordinates and assignment of k-constants*

We want simple formulae for r_1', u_1' (or l_1, as given by (56), since we are only working to first order) and c_1, these quantities being defined by an extension of the notation introduced in (147).

Now $\delta r'$ is given by (57), where δa, δe and δM are given by (138), (139) and (146). Evaluation gives

$$r_1' = \tfrac{1}{6}\bar{a}\{\bar{f}\bar{q}^2\cos 2\bar{u}+\bar{h}\bar{e}^2[2-(\bar{r}/\bar{p})(5-\cos 2\bar{v})]-6(\bar{r}/\bar{p})(\bar{h}+\tfrac{3}{4}\bar{f}\bar{e}^2\cos 2\bar{\varpi}-k_a) + \tfrac{3}{4}\bar{e}\cos\bar{v}(4\bar{h}+3\bar{f}\cos 2\bar{\varpi}-k_e)-\tfrac{3}{4}\bar{e}\sin\bar{v}(3\bar{f}\sin 2\bar{\varpi}+k_M)\}. \tag{159}$$

Clearly, the simplest formula results from taking

$$k_a = \bar{h}+\tfrac{3}{4}\bar{f}\bar{e}^2\cos 2\bar{\varpi}, \tag{160}$$

$$k_e = 4\bar{h}+3\bar{f}\cos 2\bar{\varpi} \tag{161}$$

and
$$k_M = -3\bar{f}\sin 2\bar{\varpi}, \tag{162}$$

since we then get
$$r_1' = \tfrac{1}{6}\bar{a}\{\bar{f}\bar{q}^2\cos 2\bar{u}+\bar{h}\bar{e}^2[2-(\bar{r}/\bar{p})(5-\cos 2\bar{v})]\}. \tag{163}$$

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For u_1' and c_1 we get terms involving v_M $(=v-M)$, and it is convenient to use the function of v and e defined by

$$V(v,e) = -\tfrac{4}{3}e^{-2}(v-M-2e\sin v), \tag{164}$$

i.e. $V(v,e) = \sin 2v + O(e)$, by (39). Then u_1' is given by (58), where δe, $\delta\Omega$, $\delta\omega$ and δM are given by (139), (141), (144) and (146). Evaluation gives

$$\begin{aligned} u_1' = \tfrac{1}{12}\{&\bar{f}[\sin 2\bar{u} + 4\bar{e}\sin(\bar{u}+\bar{\omega})] + 2\bar{h}(\bar{e}/\bar{q}^2)[4(3-4\bar{e}^2)\sin\bar{v} - \bar{e}\sin 2\bar{v} - \tfrac{9}{2}\bar{e}\bar{q}^2 V(\bar{v},\bar{e})] \\ &+ \tfrac{3}{2}(3\bar{f}\sin 2\bar{\omega} + k_\omega) + 6(1-\bar{f})\,k_\Omega - \tfrac{3}{2}\bar{e}\bar{q}^{-2}\sin v(2+\bar{e}\cos\bar{v})(4\bar{h} + 3\bar{f}\cos 2\bar{\omega} - k_e) \\ &- \tfrac{3}{2}\bar{q}^{-2}(1+\bar{e}\cos\bar{v})^2(3\bar{f}\sin 2\bar{\omega} + k_M)\}. \end{aligned} \tag{165}$$

This has been expressed so as to suit k_e and k_M, as given by (161) and (162); clearly we also want

$$k_\Omega = 0 \tag{166}$$

and

$$k_\omega = -3\bar{f}\sin 2\bar{\omega}, \tag{167}$$

since (165) then simplifies to

$$u_1' = \tfrac{1}{12}\{\bar{f}[\sin 2\bar{u} + 4\bar{e}\sin(\bar{u}+\bar{\omega})] + 2\bar{h}(\bar{e}/\bar{q}^2)[4(3-4\bar{e}^2)\sin\bar{v} - \bar{e}\sin 2\bar{v} - \tfrac{9}{2}\bar{e}\bar{q}^2 V(\bar{v},\bar{e})]\}. \tag{168}$$

Finally, c_1 is given by (59), where δi and $\delta\Omega$ are given by (140) and (141), so that evaluation gives

$$c_1 = \tfrac{1}{3}r\sin 2\bar{\imath}\{\bar{e}[2\sin(\bar{u}+\bar{v}) - 3\sin\bar{\omega} - \tfrac{9}{8}\bar{e}\,V(\bar{v},\bar{e})\cos\bar{u}] - \tfrac{3}{4}[(1-k_i)\sin\bar{u} + k_\Omega\cos\bar{u}]\}. \tag{169}$$

By taking

$$k_i = 1 \tag{170}$$

with (166), we obtain the simple formula

$$c_1 = \tfrac{1}{3}r\bar{e}\sin 2\bar{\imath}\{2\sin(\bar{u}+\bar{v}) - 3\sin\bar{\omega} - \tfrac{9}{8}\bar{e}\,V(\bar{v},\bar{e})\cos\bar{u}\}. \tag{171}$$

Equations (163), (168) and (171) give untruncated first-order expressions for the short-periodic perturbations in coordinates. If $O(e^2)$ terms can be neglected they reduce simply to

$$r_1' = \tfrac{1}{6}\bar{a}\bar{f}\cos 2\bar{u}, \tag{172}$$

$$u_1' = \tfrac{1}{12}\{\bar{f}\sin 2\bar{u} + 4\bar{e}[6\bar{h}\sin\bar{v} + \bar{f}\sin(\bar{u}+\bar{\omega})]\} \tag{173}$$

and

$$c_1 = \tfrac{1}{3}r\bar{e}\sin 2\bar{\imath}[2\sin(\bar{u}+\bar{v}) - 3\sin\bar{\omega}]. \tag{174}$$

Instead of u_1', we may use l_1, defined in accordance with (56), to get three expressions that are dimensionally compatible, since the analysis here is only first order. Thus (173) gives

$$l_1 = \tfrac{1}{12}r\{\bar{f}\sin 2\bar{u} + 4\bar{e}[6\bar{h}\sin\bar{v} + \bar{f}\sin(\bar{u}+\bar{\omega})]\}. \tag{175}$$

Equations (160), (161), (170), (166), (167) and (162) give 'preferred values' for k_a, k_e, k_i, k_Ω, k_ω and k_M respectively – they will be referred to as such in the remainder of the paper. From (1) and (2) we get, equivalent to (161) and (167),

$$k_\xi = (4\bar{h} + 3\bar{f})\cos\bar{\omega} \tag{176}$$

and

$$k_\eta = (4\bar{h} - 3\bar{f})\sin\bar{\omega}. \tag{177}$$

Preferred values have thus been given for all the constants except k_n, introduced in (154). The value of k_n depends on whether or not it is regarded as essential to preserve the familiar Kepler-third-law relation, as already discussed in § 4.2. If we do want to preserve this relation, then it follows at once from (148) and (154) that we must set $k_n = k_a$. If we want to assign k_a and k_n

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independently, on the other hand, then the Kepler relation must be replaced by the more general formula

$$\bar{n}^2\bar{a}^3 = \mu[1-3\bar{K}\bar{q}^{-2}(k_a-k_n)]. \tag{178}$$

It has been remarked that 'it may seem natural just to take $\bar{n} = n'$' (since $n' = \dot{M}$), and it follows from (136) and (138) that for $\bar{a} = a'$,

$$k_a = \tfrac{1}{3}\bar{h}(2+3\bar{e}^2)+\tfrac{3}{4}\bar{f}\bar{e}^2\cos 2\bar{\omega}, \tag{179}$$

whence (for $\bar{n} = n'$)

$$k_n = \tfrac{1}{3}\bar{h}(2+3\bar{e}^2)+\tfrac{3}{4}\bar{f}\bar{e}^2\cos 2\bar{\omega}. \tag{180}$$

But the preferred value of k_a is given by (160), corresponding to an $\bar{a}$ such that

$$\bar{a} = a'-\tfrac{1}{3}\bar{K}\bar{a}\bar{h}\bar{q}^{-2}(1-3\bar{e}^2). \tag{181}$$

In view of this it would be sensible practice to combine the k_a given by (160) with the k_n given by (180), such that (178) gives

$$\bar{n}^2\bar{a}^3 = \mu\{1-\bar{K}\bar{h}\bar{q}^{-2}(1-3\bar{e}^2)\}. \tag{182}$$

Unfortunately, the commonly used 'Kozai semi-major axis' (from its use by Kozai (1959)) differs slightly from the 'preferred' $\bar{a}$, since it is equivalent to the k_a given by

$$k_a = \tfrac{1}{3}\bar{h}(2+3\bar{e}^2+\bar{q}^3)+\tfrac{3}{4}\bar{f}\bar{e}^2\cos 2\bar{\omega}, \tag{183}$$

combining which with the k_n given by (180) gives Kozai's version of the Kepler third law, namely

$$\bar{n}^2\bar{a}^3 = \mu(1-\bar{K}\bar{q}\bar{h}). \tag{184}$$

The semi-major axis of the Brouwer (1959) theory, on the other hand, is our a', with k_a given by (179). Finally, an unbiased (with respect to t) mean semi-major axis is given by

$$k_a = \tfrac{1}{3}\bar{h}(2+3\bar{e}^2-2\bar{q}^3)+\tfrac{3}{4}\bar{f}\bar{e}^2\cos 2\bar{\omega}, \tag{185}$$

and this is the k_a that is effectively used by Berger & Walch (1977) and Kinoshita (1977). It will be observed that the '$\bar{f}$-component' of k_a is the same in all four expressions given, namely (160), (179), (183) and (185); as we shall see in § 9.2, this is essential if there is to be no second-order long-periodic variation in the semi-major axis – we need the $\bar{f}$-component of k_n to be specifically as given by (180) for $\bar{n}$ to be constant enough to induce a purely secular effect in M.

In proceeding with the development of second-order e-free terms, however, there is a good reason for *not* using the value of k_n given by (180) (such that $\bar{n} = n'$). In fact we want

$$\bar{n} = \dot{\bar{U}} \quad (= n'+\dot{\bar{\omega}}, \text{ by } (132)), \tag{186}$$

the significance of this value being that § 7 derives the e-free J_2^2 terms on the assumption that the relation

$$2\bar{n}\int(\cos 2\bar{U}, \sin 2\bar{U})\,\mathrm{d}t = (\sin 2\bar{U}, -\cos 2\bar{U}) \tag{187}$$

has been used in the first-order analysis. Since $\dot{\bar{\omega}}$ is given by (129), and the general relation connecting $\bar{n}$, k_n and n', if we neglect $O(\bar{e}^2)$ terms, is

$$\bar{n} = n'+\bar{K}\bar{n}(\tfrac{3}{2}k_n-\bar{h}), \tag{188}$$

we must have

$$k_n = \tfrac{2}{3}(3-4\bar{f})+O(\bar{e}^2), \tag{189}$$

from which (178) gives (with our preferred $\bar{a}$)

$$\bar{n}^2\bar{a}^3 = \mu[1+\tfrac{1}{2}\bar{K}(6-7\bar{f})]+O(\bar{K}\bar{e}^2). \tag{190}$$

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If, on the other hand, k_n is not limited to the value given by (189), then (132), (129) and (188) lead to the general relation

$$\dot{\bar{U}} - \bar{n} = \tfrac{1}{2}\bar{K}\bar{n}(6 - 8\bar{f} - 3k_n) + O(\bar{K}\bar{e}^2). \tag{191}$$

For completeness, it is worth giving the set of k_ξ corresponding to the theory of Kozai (1959), and also the set corresponding to unbiased mean elements, as used by Berger & Walch (1977) and Kinoshita (1977), in particular. In addition to k_a, k_Ω, k_ω, k_M and k_n, given by (183), (166), (167), (162) and (180) respectively, the Kozai theory requires

$$k_e = \tfrac{4}{3}\bar{h}\{5 + 2\bar{q}^2/(1+\bar{q})\} + 3\bar{f}\cos 2\bar{\omega} \tag{192}$$

and

$$k_i = 0. \tag{193}$$

For unbiased mean elements, on the other hand, we require, in addition to k_a (and also k_n) given by (185),

$$k_e = \tfrac{1}{3}\{4\bar{h}[5 + 2\bar{q}^2/(1+\bar{q})] + \bar{f}[9 - 4\bar{q}^2(1+2\bar{q})/(1+\bar{q})^2]\cos 2\bar{\omega}\}, \tag{194}$$

$$k_i = \tfrac{1}{3}\bar{e}^2\{(1+2\bar{q})/(1+\bar{q})^2\}\cos 2\bar{\omega}, \tag{195}$$

$$k_\Omega = \tfrac{1}{3}\bar{e}^2\{(1+2\bar{q})/(1+\bar{q})^2\}\sin 2\bar{\omega}, \tag{196}$$

$$k_\omega = -\tfrac{1}{3}\{3\bar{f} + 4(1+2\bar{q})(\bar{f} + \bar{e}^2 - 2\bar{e}^2\bar{f})/(1+\bar{q})^2\}\sin 2\bar{\omega} \tag{197}$$

and

$$k_M = -\tfrac{1}{3}\bar{f}\{3 + 2(2+\bar{e}^2)(1+2\bar{q})/(1+\bar{q})^2\}\sin 2\bar{\omega}. \tag{198}$$

It will be seen that mean inclination has almost always been defined such that k_i is at most $O(\bar{e}^2)$. However, the advantage of taking $k_i = 1$ was effectively recognized by King-Hele & Gilmore (1957), who based their mean inclination on the maximum north and south latitude reached by the satellite. The relation between this mean inclination and the osculating one was given by Message (1960).

7. J_2^2 Perturbations in osculating elements for orbits of low eccentricity

For any element, ζ, the secular rate of change may be expressed in the form

$$\dot{\bar{\zeta}} = \sum_{j=1}^{\infty} \bar{K}^j \dot{\bar{\zeta}}_j, \tag{199}$$

and the short-periodic perturbation in the form

$$\delta\zeta = \sum_{j=1}^{\infty} \bar{K}^j \zeta_j \tag{200}$$

(by extending the notation expressed by (147)). We are now concerned with formulae for $\dot{\bar{\zeta}}_2$ and ζ_2, but will be neglecting $O(e)$ contributions.

The basic idea is to 'bootstrap' on the first-order solution, substituting it on the right-hand sides of the planetary equations and re-integrating. The planetary equations must be taken to $O(e)$ in this process, since

$$Ke = K(\bar{e} + \delta e),$$

where

$$\delta e = \bar{K}e_1 + O(\bar{K}^2),$$

and hence Ke terms lead to $\bar{K}^2$ terms without the factor $\bar{e}$. A special procedure, free of further integration, is possible for $\zeta = a$, and this is developed first. It may be compared with the rederivation of a_2 by the general procedure.

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To keep results as general as possible, they will be quoted with unevaluated k-constants, as well as with the preferred values of § 6.3.

7.1. *Perturbation in a (special method)*

We want a_2, such that

$$a = a' + \bar{K}\bar{a}\bar{q}^{-2}(\bar{p}/\bar{r})^3(\bar{f}\cos 2\bar{u} + \tfrac{2}{3}\bar{h}) + \bar{K}^2 a_2, \tag{201}$$

a suitable starting point being the exact (136), with (cf. (188))

$$a' = \bar{a} + \bar{K}\bar{a}(k_a - \tfrac{2}{3}\bar{h}); \tag{202}$$

also $\bar{p}/\bar{r}$ can be eliminated on the basis of (48).

On comparing (136) with (201) it follows that there will be five sources of terms in a_2: (i) the 'a-variation', lost on replacing K by $\bar{K}$; (ii) the generalization from a' to $\bar{a}$, these being connected by (202); (iii) the perturbation in $(p/r)^3$; (iv) the variation in i, which affects f and h; and (v) the perturbation in u. The contributions from these sources are as follows, if we neglect terms that are $O(e)$ in a_2:

(i) $\quad -2\bar{a}(\bar{f}\cos 2\bar{u} + \tfrac{2}{3}\bar{h})(\bar{f}\cos 2\bar{u} + k_a)$;

(ii) $\quad \bar{a}(\bar{f}\cos 2\bar{u} + \tfrac{2}{3}\bar{h})(k_a - \tfrac{2}{3}\bar{h})$;

(iii) $\quad 3\bar{a}(\bar{f}\cos 2\bar{u} + \tfrac{2}{3}\bar{h})(\bar{h} + \tfrac{5}{6}\bar{f}\cos 2\bar{u})$;

(iv) $\quad \bar{a}\bar{f}(1-\bar{f})(\cos 2\bar{u} - 1)(\cos 2\bar{u} + k_i)$;

(v) $\quad \tfrac{1}{6}\bar{a}\bar{f}(6-7\bar{f})\sin^2 2\bar{u}$.

Summation of the contributions gives

$$a_2 = \tfrac{1}{3}\bar{a}\{\bar{f}^2\cos 4\bar{u} + \bar{f}[5 - 9\bar{f} - 3k_a + 3k_i(1-\bar{f})]\cos 2\bar{u} + \tfrac{1}{3}[14 - 33\bar{f} + 24\bar{f}^2 - 6k_a\bar{h} - 9k_i\bar{f}(1-\bar{f})]\}. \tag{203}$$

The constant term here is of no great general importance, and in the next section an arbitrary constant k_{2a} is introduced, by equation (209), in analogy with k_a. However, comparison of (203) and (209) enables us to obtain the expression for k_{2a} appropriate to an interpretation of $\bar{a}$ as a', i.e. to obtain the second-order relation of a to a'. We set k_a to $\tfrac{2}{3}\bar{h}$, therefore, and obtain

$$k_{2a} = \tfrac{1}{9}[(10 - 21\bar{f} + 15\bar{f}^2) - 9k_i\bar{f}(1-\bar{f})]. \tag{204}$$

Applications for this formula are found in §§ 7.9 and 9.7.

7.2. *Perturbation in a (general method)*

The starting point is the exact equation

$$\dot{a} = -\frac{Kna}{2q^5}\left(\frac{p}{r}\right)^4\{4f\sin 2u + e[4h\sin v - f\sin(u+\omega) + 5f\sin(2u+v)]\}, \tag{205}$$

from which it follows, by (48), (40) and (41), that, to $O(e)$,

$$\dot{a} = -\tfrac{1}{2}Kna\{4f\sin 2U + e[4h\sin v - f\sin(u+\omega) + 21f\sin(2u+v)]\}. \tag{206}$$

The object of replacing $\sin 2u$ by $\sin 2U$ in the main term is to permit first-order integration with respect to time (as opposed to changing the integration variable to true anomaly, by invoking (105)). On using (187), the *first-order* solution then follows at once in the form

$$\delta a = \tfrac{1}{2}\bar{K}\bar{a}\{2\bar{f}\cos 2\bar{U} + \bar{e}[4\bar{h}\cos\bar{v} - \bar{f}\cos(\bar{u}+\bar{\omega}) + 7\bar{f}\cos(2\bar{u}+\bar{v})]\}, \tag{207}$$

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which is equivalent to (148), with $k_a = 0$, if allowance is made, by (42), for the difference between $\cos 2\bar{U}$ and $\cos 2\bar{u}$.

It is now possible to expand $\dot{a}$, as given by (206), in terms of the first-order reference solutions for a, e, i, ω, U and n. Thus there are six second-order contributions to the total $\dot{a}$, and they may be denoted by $D_a\dot{a}$, $D_e\dot{a}$, etc., where

$$D_\zeta \dot{a} = (\partial\dot{a}/\partial\zeta)\, D\zeta.$$

Remembering that K is a function of a, and that we are neglecting $O(K^2e)$ perturbations, we obtain

$$D_a\dot{a} = -K\dot{a}(f\cos 2u + k_a),$$

$$D_e\dot{a} = -\tfrac{1}{24}K^2na[4h\sin v - f\sin(u+\omega) + 21f\sin(2u+v)] \times[12h\cos v + 3f\cos(u+\omega) + 7f\cos(2u+v)],$$

$$D_i\dot{a} = K\dot{a}(1-f)(\cos 2u + k_i),$$

$$D_\omega\dot{a} = \tfrac{1}{24}K^2na[4h\cos v + f\cos(u+\omega) + 21f\cos(2u+v)] \times[12h\sin v - 3f\sin(u+\omega) + 7f\sin(2u+v)],$$

$$D_U\dot{a} = -\tfrac{1}{2}K\dot{a}(2-5f)\cos 2u$$

and

$$D_n\dot{a} = -\tfrac{3}{2}K\dot{a}(f\cos 2u + k_n).$$

It is immaterial whether the quantities on the right-hand sides are regarded as osculating or mean, so 'bars' are omitted for convenience. The value of k_n must be taken from (189), however, to validate the use of (187) in deriving (207).

It now follows, after some tedious reduction, that the total second-order contribution to $\dot{a}$ is given by

$$D\dot{a} = -\tfrac{2}{3}K^2naf\{2f\sin 4u + [5 - 9f - 3k_a + 3k_i(1-f)]\sin 2u\}. \tag{208}$$

The formula for second-order δa follows immediately, and we may write

$$a_2 = \tfrac{1}{3}\bar{a}\{\bar{f}^2\cos 4\bar{u} + \bar{f}[5 - 9\bar{f} - 3k_a + 3k_i(1-\bar{f})]\cos 2\bar{u} + 3k_{2a}\}, \tag{209}$$

where it has been convenient to introduce the second-order constant k_{2a}. Clearly (209) is in agreement with (203).

Taking the preferred values of $\bar{h}$ and 1 for k_a and k_i respectively, we get

$$a_2 = \tfrac{1}{3}\bar{a}(\bar{f}^2\cos 4\bar{u} + 5\bar{f}\bar{h}\cos 2\bar{u} + 3k_{2a}), \tag{210}$$

where the preferred value of k_{2a} is given, as we shall see in § 8.1, by

$$k_{2a} = \bar{h}^2 - \tfrac{1}{72}\bar{f}^2,$$

i.e. by

$$k_{2a} = 1 - 3\bar{f} + \tfrac{161}{72}\bar{f}^2. \tag{211}$$

7.3. *Perturbation in i*

The starting point is the exact equation

$$\dot{i} = -\frac{Kn\sin 2i}{2q^3}\left(\frac{p}{r}\right)^3\sin 2u, \tag{212}$$

which to $O(e)$ gives

$$\dot{i} = -\tfrac{1}{4}Kn\sin 2i\{2\sin 2U - e[\sin(u+\omega) - 7\sin(2u+v)]\}. \tag{213}$$

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The first-order solution follows at once, in the form

$$\delta i = \tfrac{1}{12}\bar{K}\sin 2\bar{\imath}\{3\cos 2\bar{U} - \bar{e}[3\cos(\bar{u}+\bar{\omega}) - 7\cos(2\bar{u}+\bar{v})]\}, \tag{214}$$

equivalent to (149) (with $k_i = 0$).

We expand $\dot{i}$, as given by (213), in terms of the first-order reference solutions for a, e, etc., just as in §7.2, obtaining six second-order contributions to $D\dot{i}$, namely

$$D_a\dot{i} = -2K\dot{i}(f\cos 2u + k_a),$$
$$D_e\dot{i} = \tfrac{1}{48}K^2 n\sin 2i[\sin(u+\omega) - 7\sin(2u+v)][12h\cos v + 3f\cos(u+\omega) + 7f\cos(2u+v)],$$
$$D_i\dot{i} = \tfrac{1}{2}K\dot{i}\cos 2i(\cos 2u + k_i),$$
$$D_\omega\dot{i} = \tfrac{1}{48}K^2 n\sin 2i[\cos(u+\omega) + 7\cos(2u+v)][12h\sin v - 3f\sin(u+\omega) + 7f\sin(2u+v)],$$
$$D_U\dot{i} = -\tfrac{1}{2}K\dot{i}(2-5f)\cos 2u$$

and

$$D_n\dot{i} = -\tfrac{3}{2}K\dot{i}(f\cos 2u + k_n).$$

Tedious reduction, as for $D\dot{a}$, leads to the total $D\dot{i}$ given by

$$D\dot{i} = \tfrac{1}{24}K^2 n\sin 2i\{(3+5f)\sin 4u + 6[f + 4k_a - k_i(1-2f)]\sin 2u\}, \tag{215}$$

from which the formula for second-order δi follows immediately; thus

$$i_2 = -\tfrac{1}{96}\sin 2\bar{\imath}\{(3+5\bar{f})\cos 4\bar{u} + 12[\bar{f} + 4k_a - k_i(1-2\bar{f})]\cos 2\bar{u} + k_{2i}\}, \tag{216}$$

with a convenient second-order constant introduced.

With the preferred values of k_a and k_i, we get

$$i_2 = -\tfrac{1}{96}\sin 2\bar{\imath}[(3+5\bar{f})\cos 4\bar{u} + 36(1-\bar{f})\cos 2\bar{u} + k_{2i}], \tag{217}$$

where the preferred value of k_{2i} is given (as we shall see in §8.3) by

$$k_{2i} = 33 - 49\bar{f}. \tag{218}$$

7.4. *Perturbation in Ω*

The exact starting point is the exact equation

$$\dot{\Omega} = -\frac{Kn\cos i}{q^3}\left(\frac{p}{r}\right)^3(1-\cos 2u), \tag{219}$$

which to $O(e)$ gives

$$\dot{\Omega} = -\tfrac{1}{2}Kn\cos i\{2(1-\cos 2U) + e[6\cos v + \cos(u+\omega) - 7\cos(2u+v)]\}. \tag{220}$$

The first-order solution follows at once, and is composed of a secular term, with $\dot{\bar{\Omega}}$ given by (126), and a periodic term in the form

$$\delta\Omega = \tfrac{1}{6}\bar{K}\cos\bar{\imath}\{3\sin 2\bar{U} - \bar{e}[18\sin\bar{v} + 3\sin(\bar{u}+\bar{\omega}) - 7\sin(2\bar{u}+\bar{v})]\}, \tag{221}$$

equivalent to (150) (with $k_\Omega = 0$).

On expanding $\dot{\Omega}$ relative to the first-order reference solutions for a, e, etc., we obtain the following six second-order contributions to $D\dot{\Omega}$:

$$D_a\dot{\Omega} = -2K\dot{\Omega}(f\cos 2u + k_a),$$
$$D_e\dot{\Omega} = -\tfrac{1}{24}K^2 n\cos i[6\cos v + \cos(u+\omega) - 7\cos(2u+v)]$$
$$\times[12h\cos v + 3f\cos(u+\omega) + 7f\cos(2u+v)],$$

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$$\mathrm{D}_i\dot{\Omega} = -\tfrac{1}{2}K\dot{\Omega}f(\cos 2u + k_i),$$

$$\mathrm{D}_\omega\dot{\Omega} = -\tfrac{1}{24}K^2 n\cos i[6\sin v - \sin(u+\omega) - 7\sin(2u+v)] \times [12h\sin v - 3f\sin(u+\omega) + 7f\sin(2u+v)],$$

$$\mathrm{D}_U\dot{\Omega} = -K\dot{\Omega}(2-5f)\cos^2 u$$

and

$$\mathrm{D}_n\dot{\Omega} = -\tfrac{3}{2}K\dot{\Omega}(f\cos 2u + k_n).$$

The total $\mathrm{D}\dot{\Omega}$ is given by

$$\mathrm{D}\dot{\Omega} = -\tfrac{1}{6}K^2 n\cos i\{(3+f)\cos 4u - 3(2f - 4k_a - k_i f)\cos 2u + (15 - 19f - 12k_a - 3k_i f - 9k_n)\}; \tag{222}$$

thus there is a secular component of the second-order $\mathrm{D}\Omega$, and k_n has been left in (222) because the value required depends on the chosen $\bar{n}$, which appears explicitly in $\dot{\bar{\Omega}}$ (though not in Ω_2 – we have seen in § 6.3 that a mandatory k_n is associated with periodic components). We get

$$\dot{\bar{\Omega}}_2 = -\tfrac{1}{6}\bar{n}\cos\bar{\imath}(15 - 19\bar{f} - 12k_a - 3k_i\bar{f} - 9k_n) \tag{223}$$

and

$$\Omega_2 = -\tfrac{1}{24}\cos\bar{\imath}[(3+\bar{f})\sin 4\bar{u} - 6(2\bar{f} - 4k_a - k_i\bar{f})\sin 2\bar{u}], \tag{224}$$

no $k_{2\Omega}$ constant being required. (It would be the coefficient of $\sin 0u$!)

With the preferred values of k_a and k_i, we get

$$\dot{\bar{\Omega}}_2 = -\tfrac{1}{6}\bar{n}\cos\bar{\imath}(3 - 4f - 9k_n) \tag{225}$$

and

$$\Omega_2 = -\tfrac{1}{24}\cos\bar{\imath}[(3+\bar{f})\sin 4\bar{u} + 6(4 - 7\bar{f})\sin 2\bar{u}]. \tag{226}$$

With k_n given by (189) (cf. (327) if k_n is taken from (180)) (226) gives

$$\dot{\bar{\Omega}}_2 = \tfrac{5}{6}\bar{n}\cos\bar{\imath}(3 - 4\bar{f}). \tag{227}$$

7.5. *Perturbation in* ξ

The exact starting equation is

$$\dot{\xi} = -\frac{Kn}{4q^3}\left(\frac{p}{r}\right)^3\{(4h+f)\sin u + 7f\sin 3u + \tfrac{1}{2}e[3(4-5f)\sin\omega + (8-7f)\sin(u+v) - (4-13f)\sin(2u+\omega) + 5f\sin(3u+v)]\}, \tag{228}$$

which to $O(e)$ gives

$$\dot{\xi} = -\tfrac{1}{4}Kn\{(4h+f)\sin U + 7f\sin 3U + 2e[(4-5f)\sin\omega + (7-8f)\sin(u+v) - (1+2f)\sin(2u+\omega) + 17f\sin(3u+v)]\}. \tag{229}$$

The first-order solution follows at once, in a form consisting of a secular term, with $\dot{\bar{\xi}}_1$ given by (129) multiplied by $-\bar{e}\sin\bar{\omega}$, and a periodic term

$$\delta\xi = \tfrac{1}{24}\bar{K}\{6(4\bar{h}+\bar{f})\cos\bar{U} + 14\bar{f}\cos 3\bar{U} + 3\bar{e}[2(7-8\bar{f})\cos(\bar{u}+\bar{v}) - 2(1+2\bar{f})\cos(2\bar{u}+\bar{\omega}) + 17\bar{f}\cos(3\bar{u}+\bar{v})]\}, \tag{230}$$

equivalent to (151) with k_ξ taken as $2(4\bar{h}+\bar{f})\cos\bar{\omega}$.

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On expanding $\dot{\xi}$ relative to the usual first-order reference solutions, we obtain

$$D_a\dot{\xi} = -2K\dot{\xi}(f\cos 2u + k_a),$$

$$D_e\dot{\xi} = -\tfrac{1}{24}K^2n[(4-5f)\sin\omega + (7-8f)\sin(u+v) - (1+2f)\sin(2u+\omega) + 17f\sin(3u+v)]$$
$$\times[12h\cos v + 3f\cos(u+\omega) + 7f\cos(2u+v)],$$

$$D_i\dot{\xi} = \tfrac{1}{4}K^2nf(1-f)(5\sin u - 7\sin 3u)(\cos 2u + k_i),$$

$$D_\omega\dot{\xi} = -\tfrac{1}{24}K^2n[(4-5f)\cos\omega - (7-8f)\cos(u+v) - (1+2f)\cos(2u+\omega) - 17f\cos(3u+v)]$$
$$\times[12h\sin v - 3f\sin(u+\omega) + 7f\sin(2u+v)],$$

$$D_U\dot{\xi} = \tfrac{1}{16}K^2n(2-5f)[(4h+f)\cos u + 21f\cos 3u]\sin 2u$$

and

$$D_n\dot{\xi} = -\tfrac{3}{2}K\dot{\xi}(f\cos 2u + k_n).$$

The total $D\dot{\xi}$ is given by

$$D\dot{\xi} = \tfrac{1}{96}K^2n\{5f(14-17f)\sin 5u + [72 - 346f + 449f^2 + 336fk_a - 168k_if(1-f)]\sin 3u$$
$$-2[108 - 200f + 77f^2 - 24k_a(4-5f) - 60k_if(1-f)]\sin u\}, \tag{231}$$

whence

$$\xi_2 = -\tfrac{1}{288}\{3\bar{f}(14-17\bar{f})\cos 5\bar{u} + [72 - 346\bar{f} + 449\bar{f}^2 + 336\bar{f}k_a - 168k_i\bar{f}(1-\bar{f})]\cos 3\bar{u}$$
$$-6[108 - 200\bar{f} + 77\bar{f}^2 - 24k_a(4-5\bar{f}) - 60k_i\bar{f}(1-\bar{f})]\cos\bar{u}\}, \tag{232}$$

no constant being introduced.

With the preferred values of k_a and k_i, we get

$$\xi_2 = -\tfrac{1}{288}\{3\bar{f}(14-17\bar{f})\cos 5\bar{u} + (72 - 178\bar{f} + 113\bar{f}^2)\cos 3\bar{u} - 6(12 + 4\bar{f} - 43\bar{f}^2)\cos\bar{u}\}. \tag{233}$$

7.6. *Perturbation in η*

The exact starting equation is

$$\dot{\eta} = \frac{Kn}{4q^3}\left(\frac{p}{r}\right)^3\{(4h-f)\cos u + 7f\cos 3u + \tfrac{1}{2}e[(12-13f)\cos\omega - 5f\cos(u+v)$$
$$-(4-12f)\cos(2u+\omega) + 5f\cos(3u+v)]\}, \tag{234}$$

which to $O(e)$ gives

$$\dot{\eta} = \tfrac{1}{4}Kn\{(4h-f)\cos U + 7f\cos 3U + 2e[(4-5f)\cos\omega + 5(1-2f)\cos(u+v)$$
$$-(1+2f)\cos(2u+\omega) + 17f\cos(3u+v)]\}. \tag{235}$$

The first-order solution follows at once, in a form consisting of a secular term, with $\bar{\dot{\eta}}_1$ given by (129) multiplied by $\bar{e}\cos\bar{\omega}$, and a periodic term

$$\delta\eta = \tfrac{1}{24}\bar{K}\{6(4\bar{h}-\bar{f})\sin\bar{U} + 14\bar{f}\sin 3\bar{U} + 3\bar{e}[10(1-2\bar{f})\sin(\bar{u}+\bar{v})$$
$$-2(1+2\bar{f})\sin(2\bar{u}+\bar{\omega}) + 17\bar{f}\sin(3\bar{u}+\bar{v})]\}, \tag{236}$$

equivalent to (152) with k_η taken as $2(4\bar{h}-\bar{f})\sin\bar{\omega}$.

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On expanding $\dot{\eta}$ relative to the usual first-order reference solutions, we obtain:

$$\mathrm{D}_a\dot{\eta} = -2K\dot{\eta}(f\cos 2u + k_a),$$

$$\mathrm{D}_e\dot{\eta} = \tfrac{1}{24}K^2n[(4-5f)\cos\omega + 5(1-2f)\cos(u+v) - (1+2f)\cos(2u+\omega) + 17f\cos(3u+v)] \times [12h\cos v + 3f\cos(u+\omega) + 7f\cos(2u+v)],$$

$$\mathrm{D}_i\dot{\eta} = -\tfrac{7}{4}K^2nf(1-f)(\cos u - \cos 3u)(\cos 2u + k_i),$$

$$\mathrm{D}_\omega\dot{\eta} = -\tfrac{1}{24}K^2n[(4-5f)\sin\omega - 5(1-2f)\sin(u+v) - (1+2f)\sin(2u+\omega) - 17f\sin(3u+v)] \times [12h\sin v - 3f\sin(u+\omega) + 7f\sin(2u+v)],$$

$$\mathrm{D}_U\dot{\eta} = \tfrac{1}{16}K^2n(2-5f)[(4h-f)\sin u + 21f\sin 3u]\sin 2u$$

and

$$\mathrm{D}_n\dot{\eta} = -\tfrac{3}{2}K\dot{\eta}(f\cos 2u + k_n).$$

The total $\mathrm{D}\dot{\eta}$ is given by

$$\mathrm{D}\dot{\eta} = -\tfrac{1}{96}K^2n\{5f(14-17f)\cos 5u + [72 - 310f + 395f^2 + 336fk_a - 168k_if(1-f)]\cos 3u - 2[84 - 224f + 155f^2 - 24k_a(4-7f) - 84k_if(1-f)]\cos u\}, \tag{237}$$

whence

$$\eta_2 = -\tfrac{1}{288}\{3\bar{f}(14-17\bar{f})\sin 5\bar{u} + [72 - 310\bar{f} + 395\bar{f}^2 + 336\bar{f}k_a - 168k_i\bar{f}(1-\bar{f})]\sin 3\bar{u} - 6[84 - 224\bar{f} + 155\bar{f}^2 - 24k_a(4-7\bar{f}) - 84k_i\bar{f}(1-\bar{f})]\sin\bar{u}\}, \tag{238}$$

no constant being introduced.

With the preferred values of k_a and k_i, we get

$$\eta_2 = -\tfrac{1}{288}\{3\bar{f}(14-17\bar{f})\sin 5\bar{u} + (72 - 142\bar{f} + 59\bar{f}^2)\sin 3\bar{u} + 6(12 - 4\bar{f} + 13\bar{f}^2)\sin\bar{u}\}. \tag{239}$$

7.7. *Perturbation in $\sigma+\omega$*

From the exact equations:

$$e\dot{\omega} = \frac{Kn}{4q^3}\left(\frac{p}{r}\right)^3\{4h\cos v - f\cos(u+\omega) + 7f\cos(2u+v) + \tfrac{1}{2}e[4h\cos 2v - 2(4-7f)\cos 2u + 5f\cos 2(u+v) + f\cos 2\omega + 2(6-7f)]\} \tag{240}$$

and

$$e\dot{\sigma} = -\frac{Kn}{4q^2}\left(\frac{p}{r}\right)^3\{4h\cos v - f\cos(u+\omega) + 7f\cos(2u+v) + \tfrac{1}{2}e[4h\cos 2v - 18f\cos 2u + 5f\cos 2(u+v) + f\cos 2\omega - 12h]\}, \tag{241}$$

it follows that, to $O(e)$,

$$\dot{\sigma} + \dot{\omega} = \tfrac{1}{8}Kn\{8[(3-4f) - (1-4f)\cos 2U] + e[2(38-51f)\cos v + (4-17f)\cos(u+\omega) - 7(4-17f)\cos(2u+v)]\}. \tag{242}$$

The first-order solution follows at once; it consists of a secular term, in conformity with (129) and (130), and a periodic term in the form

$$\delta\sigma + \delta\omega = -\tfrac{1}{24}\bar{K}\{12(1-4\bar{f})\sin 2\bar{U} - \bar{e}[6(38-51\bar{f})\sin\bar{v} + 3(4-17\bar{f})\sin(\bar{u}+\bar{\omega}) - 7(4-17\bar{f})\sin(2\bar{u}+\bar{v})]\}. \tag{243}$$

This is equivalent, by (42), to a result that could be added to equations (148)–(154), namely

$$\sigma_1 + \omega_1 = -\tfrac{1}{48}\{24(1-4\bar{f})\sin 2\bar{u} - \bar{e}[6(38-51\bar{f})\sin\bar{v} - 3(4-15\bar{f})\sin(\bar{u}+\bar{\omega}) - (4-23\bar{f})\sin(2\bar{u}+\bar{v})]\}. \tag{244}$$

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On expanding $\dot{\sigma}+\dot{\omega}$ relative to the first-order reference solutions, as usual, we obtain:

$$D_a(\dot{\sigma}+\dot{\omega}) = -2K(\dot{\sigma}+\dot{\omega})\,(f\cos 2u+k_a),$$

$$D_e(\dot{\sigma}+\dot{\omega}) = \tfrac{1}{96}K^2n[2(38-51f)\cos v+(4-17f)\cos(u+\omega)-7(4-17f)\cos(2u+v)]$$
$$\times[12h\cos v+3f\cos(u+\omega)+7f\cos(2u+v)],$$

$$D_i(\dot{\sigma}+\dot{\omega}) = -4K^2nf(1-f)\,(1-\cos 2u)\,(\cos 2u+k_i),$$

$$D_\omega(\dot{\sigma}+\dot{\omega}) = \tfrac{1}{96}K^2n[2(38-51f)\sin v-(4-17f)\sin(u+\omega)-7(4-17f)\sin(2u+v)]$$
$$\times[12h\sin v-3f\sin(u+\omega)+7f\sin(2u+v)],$$

$$D_U(\dot{\sigma}+\dot{\omega}) = -\tfrac{1}{2}K^2n(1-4f)\,(2-5f)\sin^2 2u$$

and

$$D_n(\dot{\sigma}+\dot{\omega}) = -\tfrac{3}{2}K(\dot{\sigma}+\dot{\omega})\,(f\cos 2u+k_n).$$

The total $D(\dot{\sigma}+\dot{\omega})$ is given by

$$D(\dot{\sigma}+\dot{\omega}) = \tfrac{1}{48}K^2n\{(24-4f-73f^2)\cos 4u-4[f(64-51f)-24k_a(1-4f)-48k_if(1-f)]\cos 2u$$
$$+[(432-1052f+637f^2)-24(3-4f)\,(4k_a+3k_n)-192k_if(1-f)]\}, \tag{245}$$

where a term in k_n has been retained in the secular component for the same reason as with $D\dot{\Omega}$. We get

$$(\dot{\bar{\sigma}}+\dot{\bar{\omega}})_2 = \tfrac{1}{48}\bar{n}[(432-1052\bar{f}+637\bar{f}^2)-24(3-4\bar{f})\,(4k_a+3k_n)-192k_i\bar{f}(1-\bar{f})] \tag{246}$$

and

$$(\sigma+\omega)_2 = \tfrac{1}{192}\{(24-4\bar{f}-73\bar{f}^2)\sin 4\bar{u}-8[\bar{f}(64-51\bar{f})-24k_a(1-4\bar{f})-48k_i\bar{f}(1-\bar{f})]\sin 2\bar{u}\}. \tag{247}$$

With the preferred values of k_a and k_i, we get

$$(\dot{\bar{\sigma}}+\dot{\bar{\omega}})_2 = \tfrac{1}{48}\bar{n}[(144-428\bar{f}+253\bar{f}^2)-72k_n(3-4\bar{f})] \tag{248}$$

and
$$(\sigma+\omega)_2 = \tfrac{1}{192}\{(24-4\bar{f}-73\bar{f}^2)\sin 4\bar{u}+8(24-148\bar{f}+147\bar{f}^2)\sin 2\bar{u}\}, \tag{249}$$

and with k_n given by (189) we get

$$(\dot{\bar{\sigma}}+\dot{\bar{\omega}})_2 = -\tfrac{1}{48}\bar{n}(288-724\bar{f}+515\bar{f}^2). \tag{250}$$

7.8. *Perturbation in n and in* $\int n\,dt$

To be able to analyse the perturbation in U, as we shall in § 7.9, we need a formula for the perturbation in $\int_0^t n\,dt$, to combine with $D(\sigma+\omega)$, after (12) and (3). Thus we first need the formula for n_2 that is associated with the perturbation in n.

The perturbation in n could be obtained by the usual method, involving integration, but it is more direct to obtain it from the perturbation in a, by making use of the fundamental Kepler relation (4). This is possible even though, because the k-constants for n and a are independent, $\bar{n}$ and $\bar{a}$ will not in general satisfy the relation themselves. We introduce $\hat{\mu}_1$ and $\hat{\mu}_2$, therefore, dependent only on $\bar{f}$ (so long as we work only to first order in e), such that

$$\bar{n}^2\bar{a}^3 = \mu(1+\bar{K}\hat{\mu}_1+\bar{K}^2\hat{\mu}_2). \tag{251}$$

This is equivalent to the relation

$$n^2a^3 = \bar{n}^2\bar{a}^3(1+\bar{K}\hat{\mu}_1+\bar{K}^2\hat{\mu}_2)^{-1}, \tag{252}$$

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where a strong reason for introducing the relation in this form, rather than reciprocally, is that the final expression for $\hat{\mu}_2$ – see (278) – becomes very simple; reciprocally, the relation is

$$n^2a^3 = \bar{n}^2\bar{a}^3[1-\bar{K}\hat{\mu}_1-\bar{K}^2(\hat{\mu}_2-\hat{\mu}_1^2)]. \tag{253}$$

The general first-order result, previously given by (178), may now be written in the form

$$\hat{\mu}_1 = 3(k_n-k_a), \tag{254}$$

neglecting the factor $\bar{q}^{-2}$ which contributes $O(\bar{e}^2)$ terms, and with the preferred k_a and the usual k_n this reduces (cf. (190)) to

$$\hat{\mu}_1 = \tfrac{1}{2}(6-7\bar{f}). \tag{255}$$

But $$a = \bar{a}+\bar{K}a_1+\bar{K}^2a_2$$

and $$n = \bar{n}+\bar{K}n_1+\bar{K}^2n_2,$$

from which it follows that

$$n^2a^3 = \bar{n}^2\bar{a}^3(1+\bar{K}\hat{a}_1+\bar{K}^2\hat{a}_2)^3(1+\bar{K}\hat{n}_1+\bar{K}^2\hat{n}_2)^2, \tag{256}$$

on defining $\hat{a}_1$ to be $a_1/\bar{a}$, etc., for convenience. On comparing (253) and (256) it follows that

$$\hat{\mu}_1 = -(3\hat{a}_1+2\hat{n}_1) \tag{257}$$

and $$\hat{\mu}_2 = 6\hat{a}_1^2+6\hat{a}_1\hat{n}_1+3\hat{n}_1^2-3\hat{a}_2-2\hat{n}_2. \tag{258}$$

There is nothing new in (257), which just leads to (254) again, but (258) leads to the desired formula for n_2. To see this, we start with formulae for $\hat{a}_1$, $\hat{n}_1$ and $\hat{a}_2$, which follow from (148), (154) and (209) respectively, whereupon (258) gives

$$\begin{aligned}2\hat{n}_2+\hat{\mu}_2 = \tfrac{1}{8}\{&7\bar{f}^2\cos 4\bar{u}-4\bar{f}[2(5-9\bar{f})-12k_a+6k_i(1-\bar{f})-9k_n]\cos 2\bar{u}\\ &+3(5\bar{f}^2+16k_a^2-24k_ak_n+18k_n^2-8k_{2a})\}.\end{aligned} \tag{259}$$

(It is legitimate to leave k_n in the coefficient of $\cos 2\bar{u}$, in spite of the remarks of § 6.3, because it arises through expansion of (258) and not from integration.)

But $\hat{\mu}_2$ must be independent of $\bar{u}$, so we must in consequence of (259) be able to write

$$n_2 = \tfrac{1}{16}\bar{n}\{7\bar{f}^2\cos 4\bar{u}-4\bar{f}[2(5-9\bar{f})-12k_a+6k_i(1-\bar{f})-9k_n]\cos 2\bar{u}+16k_{2n}\}, \tag{260}$$

where the introduced constant k_{2n} is such that

$$\hat{\mu}_2 = \tfrac{3}{8}(5\bar{f}^2+16k_a^2-24k_ak_n+18k_n^2-8k_{2a})-2k_{2n}. \tag{261}$$

With the preferred k_a and k_i, and the usual k_n, (260) reduces to

$$n_2 = \tfrac{1}{16}\bar{n}\{7\bar{f}^2\cos 4\bar{u}+8\bar{f}(7-9\bar{f})\cos 2\bar{u}+16k_{2n}\} \tag{262}$$

and (261) to $$\hat{\mu}_2 = \tfrac{3}{8}(40-104\bar{f}+73\bar{f}^2-8k_{2a})-2k_{2n}. \tag{263}$$

To obtain the perturbation in $\int_0^t n\,dt$, we observe that

$$n = \bar{n}+\bar{K}n_1+\bar{K}^2n_2, \tag{264}$$

where n_1 (cf. (154) and (207)) is given by

$$n_1 = -\tfrac{3}{2}\bar{n}\{\bar{f}\cos 2\bar{U}+k_n+\tfrac{1}{2}\bar{e}[4\bar{h}\cos\bar{v}-\bar{f}\cos(\bar{u}+\bar{\omega})+7\bar{f}\cos(2\bar{u}+\bar{v})]\}. \tag{265}$$

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Then writing the required integral as a sum of secular and periodic components, we have

$$\int_0^t n\,\mathrm{d}t = (\bar{n}+\bar{\delta}\bar{n})\,t + \bar{K}\int_1 + \bar{K}^2\int_2, \tag{266}$$

where (265) and (262) imply that

$$\bar{\delta}\bar{n}/\bar{n} = -\tfrac{3}{2}\bar{K}k_n + \bar{K}^2 k_{2n}. \tag{267}$$

For $\int_1$, we integrate the periodic term of (265), obtaining

$$\int_1 = -\tfrac{1}{4}\{3\bar{f}\sin 2\bar{U} + \bar{e}[12\bar{h}\sin\bar{v} - 3\bar{f}\sin(\bar{u}+\bar{\omega}) + 7\bar{f}\sin(2\bar{u}+\bar{v})]\}. \tag{268}$$

For $\int_2$, similarly, we integrate (260) to obtain (since k_n must necessarily be taken from (189) as usual)

$$\int_2 = \tfrac{1}{64}\{7\bar{f}^2\sin 4\bar{u} + 16\bar{f}[4-3\bar{f}+6k_a-3k_i(1-\bar{f})]\sin 2\bar{u}\}, \tag{269}$$

which with the preferred values of k_a and k_i (or from (262) directly) reduces to

$$\int_2 = \tfrac{1}{64}\{7\bar{f}^2\sin 4\bar{u} + 16\bar{f}(7-9\bar{f})\sin 2\bar{u})\}. \tag{270}$$

7.9. *Perturbation in U*

From (3), (12) and (266) we obtain at once, on combining (129) and (130) with (267), and (243) with (268), the first-order secular result, i.e. $\dot{\bar{U}}_1$, implied by (191), and the periodic term given by

$$U_1 = -\tfrac{1}{24}\{6(2-5\bar{f})\sin 2\bar{U} - \bar{e}[6(26-33\bar{f})\sin\bar{v} + 3(4-11\bar{f})\sin(\bar{u}+\bar{\omega}) - 7(4-11\bar{f})\sin(2\bar{u}+\bar{v})]\}, \tag{271}$$

which is equivalent to (153).

The expression for U_2 is given by combining (247) and (269); thus

$$U_2 = \tfrac{1}{48}\{(6-\bar{f}-13\bar{f}^2)\sin 4\bar{u} - 2[\bar{f}(40-33\bar{f}) - 12k_a(2-5\bar{f}) - 30k_i\bar{f}(1-\bar{f})]\sin 2\bar{u}\}. \tag{272}$$

With the preferred values of k_a and k_i,

$$U_2 = \tfrac{1}{48}\{(6-\bar{f}-13\bar{f}^2)\sin 4\bar{u} + 2(24-106\bar{f}+93\bar{f}^2)\sin 2\bar{u}\}. \tag{273}$$

Again, the general expression for $\dot{\bar{U}}_2$ is given by combining (246) with the second-order component of $\bar{\delta}\bar{n}$, given by (267); thus

$$\dot{\bar{U}}_2 = \tfrac{1}{48}\bar{n}[(432-1052\bar{f}+637\bar{f}^2) - 24(3-4\bar{f})(4k_a+3k_n) - 192k_i\bar{f}(1-\bar{f}) + 48k_{2n}]. \tag{274}$$

When $\dot{\bar{U}}_1$ is zero, through k_n having the value given by (189), it is obviously desirable for $\dot{\bar{U}}_2$ to be zero as well, and (274) indicates that this requires

$$k_{2n} = -\tfrac{1}{48}\{\bar{f}(100-131\bar{f}) - 96k_a(3-4\bar{f}) - 192k_i\bar{f}(1-\bar{f})\}. \tag{275}$$

With the preferred values of k_a and k_i, this gives

$$k_{2n} = \tfrac{1}{48}(288-724\bar{f}+515\bar{f}^2). \tag{276}$$

The associated value of $\hat{\mu}_2$ is given by (261). Taking k_{2n} as given by (275), we find

$$\hat{\mu}_2 = \tfrac{1}{12}\{(324-814\bar{f}+533\bar{f}^2) - 120k_a(3-4\bar{f}) + 72k_a^2 - 96k_i\bar{f}(1-\bar{f}) - 36k_{2a}\}, \tag{277}$$

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but with the preferred values of k_a, k_i and k_{2a} we get a tremendous simplification, to

$$\hat{\mu}_2 = \tfrac{1}{24}\bar{f}(4-19\bar{f}). \tag{278}$$

It may, however, be considered undesirable to tamper with the Kepler relation, in which case $\hat{\mu}_2$ must be zero and we cannot force $\dot{\bar{U}}_2$ to be zero. But, for (261) to give zero, we require

$$k_{2n} = \tfrac{3}{16}(5\bar{f}^2+16k_a^2-24k_a k_n+18k_n^2-8k_{2a}) \tag{279}$$

and then (274) gives

$$\dot{\bar{U}}_2 = \tfrac{1}{24}\bar{n}\{(216-526\bar{f}+341\bar{f}^2)-12(3-4\bar{f})(4k_a+3k_n)-96k_i\bar{f}(1-\bar{f}) + 72k_a^2-108k_a k_n+81k_n^2-36k_{2a}\}. \tag{280}$$

Instead of substituting our preferred k_a and the usual k_n in (279), it is instructive to set $k_a = k_n = \frac{2}{3}\bar{h}$, with k_{2a} given by (204), i.e. appropriate to $\bar{a} = a'$, since we can then interpret $\bar{n}$ as the exact constant n'. The resulting expression for k_2n is given by

$$k_{2n} = -\tfrac{1}{48}[(40-48\bar{f}-15\bar{f}^2)-72k_i\bar{f}(1-\bar{f})]; \tag{281}$$

with the preferred k_i,

$$k_{2n} = -\tfrac{1}{48}(40-120\bar{f}+57\bar{f}^2). \tag{282}$$

7.10. *Comparisons with other authors' results*

The formulae obtained have been compared in detail with the results of Bretagnon (1972), Berger & Walch (1977) and Kinoshita (1977). A full report of these comparisons was made by Gooding (1979) and will merely be summarized here.

To compare with the two French papers (Bretagnon 1972, Berger & Walch 1977), it was necessary to derive results for e_2 and ω_2 from those (in §§7.5 and 7.6) for ξ_2 and η_2 This was not a trivial matter, since e_2 and ω_2 are $O(\bar{e}^{-1})$ and $O(\bar{e}^{-2})$, respectively, whereas (effectively because of the proximity of the eccentricity singularity) results were needed to $O(\bar{e}^0)$ and $O(\bar{e}^{-1})$.

Complete agreement with the results of Berger & Walch was obtained, on the assumption that they used unbiased mean elements, with implicit first-order k_ζ given by (185) and (194)–(198), and with the second-order constants (k_{2a} and k_{2i}) both zero. However, argreement with the results of Bretagnon (1972) could only be obtained by changing the values of k_a and k_i, in an apparently arbitrary manner, as each ζ_2 in turn was checked.

To compare with Kinoshita's results it was first necessary to obtain expressions for α_1 and α_2, where $\alpha = a^{\frac{1}{2}}$ and the basic identity is

$$(\bar{\alpha}+\bar{K}\alpha_1+\bar{K}^2\alpha_2)^2 = \bar{a}+\bar{K}a_1+\bar{K}^2a_2. \tag{283}$$

We have the first-order relation

$$2\bar{\alpha}\alpha_1 = a_1 \tag{284}$$

and the second-order relation

$$\alpha_1^2+2\bar{\alpha}\alpha_2 = a_2, \tag{285}$$

which yield

$$\alpha_2 = \tfrac{1}{48}\bar{\alpha}\{5\bar{f}^2\cos 4\bar{u}+4\bar{f}[2(5-9\bar{f})-9k_a+6k_a(1-\bar{f})]\cos 2\bar{u}-3(\bar{f}^2+2k_a^2-8k_{2a})\}. \tag{286}$$

Complete agreement was obtained, but only after setting k_{2a} empirically to $\frac{1}{12}(12-30\bar{f}+25\bar{f}^2)$. It seems that this value was (effectively) used by Brouwer (1959), since this assumption also resolves a discrepancy in formulae for $\dot{\bar{M}}_2$ as given by Bretagnon (1972) and Brouwer (1959), remarked upon by Bretagnon (1972).

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8. J_2^2 Perturbations in position

To obtain perturbations in the cylindrical polar coordinates described in § 3.3 is essentially a matter of combining existing results, no further integration being required. As a simplification, we throughout set k_a, k_ξ, k_η and k_i to their preferred values, as given by (160), (176), (177) and (170), since this preference was based on the resulting simplicity in the first-order formulae for perturbations in coordinates.

It is convenient to give, in advance, some of the formulae that will be required. Thus, neglecting $O(\bar{e})$ terms in each case,

$$\xi_1 \cos\bar{u} + \eta_1 \sin\bar{u} = \bar{h} + \tfrac{5}{6}\bar{f}\cos 2\bar{u}, \tag{287}$$

$$\xi_1 \sin\bar{u} - \eta_1 \cos\bar{u} = -\tfrac{1}{3}\bar{f}\sin 2\bar{u}, \tag{288}$$

$$\xi_2 \cos\bar{u} + \eta_2 \sin\bar{u} = -\tfrac{1}{36}\bar{f}\{3(1-\bar{f})\cos 4\bar{u} - 2(10-11\bar{f})\cos 2\bar{u} - 3(1-7\bar{f})\}, \tag{289}$$

$$\xi_2 \sin\bar{u} - \eta_2 \cos\bar{u} = \tfrac{1}{144}\{3\bar{f}(10-13\bar{f})\sin 4\bar{u} + 2(36-40\bar{f}-\bar{f}^2)\sin 2\bar{u}\} \tag{290}$$

and

$$(\xi_1^2 - \eta_1^2)\sin 2\bar{u} - 2\xi_1\eta_1\cos 2\bar{u} = -\tfrac{1}{18}\bar{f}(5\bar{f}\sin 4\bar{u} + 12\bar{h}\sin 2\bar{u}). \tag{291}$$

These formulae follow at once from (151), (152), (233) and (239).

8.1. *Perturbation in r*

We require formulae for r_1 and r_2, such that

$$r = \bar{r} + \bar{K}r_1 + \bar{K}^2 r_2, \tag{292}$$

where $\bar{r}$ is derived from mean elements by application of the standard algorithm of § 3.3, while r_1 and r_2 are purely periodic.

Now (49) and (45) yield

$$r = a(1 - e\cos M + e^2 \sin^2 M) + O(e^3),$$

from which it follows that

$$\begin{aligned} r = (\bar{a} + \bar{K}a_1 + \bar{K}^2 a_2)\{1 &- (\bar{\xi} + \bar{K}\xi_1 + \bar{K}^2\xi_2)(\cos\bar{U} - \bar{K}U_1\sin\bar{U}) \\ &- (\bar{\eta} + \bar{K}\eta_1 + \bar{K}^2\eta_2)(\sin\bar{U} + \bar{K}U_1\cos\bar{U}) + [(\bar{\xi} + \bar{K}\xi_1)\sin\bar{U} - (\bar{\eta} + \bar{K}\eta_1)\cos\bar{U}]^2\}, \end{aligned} \tag{293}$$

only such terms being retained as will actually be needed.

On expanding (293) and comparing with (292), we obtain the first-order formula

$$r_1 = (\bar{r}/\bar{a})\,a_1 - \bar{a}[(\xi_1\cos\bar{U} + \eta_1\sin\bar{U}) - \bar{e}\sin\bar{M}(U_1 + 2\xi_1\sin\bar{U} - 2\eta_1\cos\bar{U})]. \tag{294}$$

The formula for r_2, similarly, is

$$\begin{aligned} r_2 = (\bar{r}/\bar{a})\,a_2 - a_1(\xi_1\cos\bar{U} + \eta_1\sin\bar{U}) &- \bar{a}[(\xi_2\cos\bar{U} + \eta_2\sin\bar{U}) \\ &- (\xi_1\sin\bar{U} - \eta_1\cos\bar{U})(\xi_1\sin\bar{U} - \eta_1\cos\bar{U} + U_1)]. \end{aligned} \tag{295}$$

We can replace $\bar{U}$ by $\bar{u}$, of course, since we are ignoring $O(\bar{K}e)$-perturbations. Then r_2 is a combination of terms given by (210), (148), (287), (289), (288) and (153), the result on reduction being

$$r_2 = -\tfrac{1}{72}\bar{a}\{\bar{f}^2\cos 4\bar{u} + 2\bar{f}(26-31\bar{f})\cos 2\bar{u} + 72\bar{h}^2 - \bar{f}^2 - 72k_{2a}\}. \tag{296}$$

Choice of k_{2a}, as anticipated by (211), now gives our final result, namely

$$r_2 = -\tfrac{1}{72}\bar{a}\bar{f}\{\bar{f}\cos 4\bar{u} + 2(26-31\bar{f})\cos 2\bar{u}\}. \tag{297}$$

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8.2. *Perturbation in u*

We require formulae for u_1 and u_2, such that

$$u = \bar{u} + \bar{K}u_1 + \bar{K}^2 u_2, \tag{298}$$

where u_1 and u_2 are purely periodic.

Now (38) yields $u = U + 2e\sin M + \frac{5}{4}e^2 \sin 2M + O(e^3)$,

from which it follows that

$$\begin{aligned} u = {} & \bar{U} + \bar{K}U_1 + \bar{K}^2 U_2 + 2(\bar{\xi} + \bar{K}\xi_1 + \bar{K}^2\xi_2)(\sin\bar{U} + \bar{K}U_1 \cos\bar{U}) \\ & - 2(\bar{\eta} + \bar{K}\eta_1 + \bar{K}^2\eta_2)(\cos\bar{U} - \bar{K}U_1 \sin\bar{U}) \\ & + \tfrac{5}{4}\{[(\bar{\xi} + \bar{K}\xi_1)^2 - (\eta + \bar{K}\eta_1)^2]\sin 2\bar{U} - 2(\bar{\xi} + \bar{K}\xi_1)(\bar{\eta} + \bar{K}\eta_1)\cos 2\bar{U}\}, \end{aligned} \tag{299}$$

with only the needed terms retained.

On expanding (299) and comparing with (298), we obtain the first-order formula

$$u_1 = U_1 + 2(\xi_1 \sin\bar{U} - \eta_1 \cos\bar{U}) + \tfrac{1}{2}\bar{e}[4U_1 \cos\bar{M} + 5\xi_1 \sin(\bar{u} + \bar{v}) - 5\eta_1 \cos(\bar{u} + \bar{v})]. \tag{300}$$

The formula for u_2, similarly, is

$$\begin{aligned} u_2 = {} & U_2 + 2(\xi_2 \sin\bar{U} - \eta_2 \cos\bar{U}) + 2U_1(\xi_1 \cos\bar{U} + \eta_1 \sin\bar{U}) \\ & + \tfrac{5}{4}[(\xi_1^2 - \eta_1^2)\sin 2\bar{U} - 2\xi_1\eta_1 \cos 2\bar{U}]. \end{aligned} \tag{301}$$

We replace $\bar{U}$ by $\bar{u}$ and obtain u_2 as a combination of terms given by (273), (290), (153), (287) and (291), the result being

$$u_2 = \tfrac{1}{144}\{(18 - 3\bar{f} - 17\bar{f}^2)\sin 4\bar{u} + 2(72 - 170\bar{f} + 97\bar{f}^2)\sin 2\bar{u}\}. \tag{302}$$

8.3. *Perturbations in r′, u′ and c*

The first-order results, given in § 6.3, were based on formulae (57)–(59), which are not valid when we proceed to second order. Thus we require a fresh start, based on the formula (54) that effectively defines the cylindrical polar coordinates r', u' and c.

Now $(x \quad y \quad z)^{\mathrm{T}}$ is given by both (50) and (54), so an exact equation for r', u' and c is

$$(r'\cos u' \quad r'\sin u' \quad c)^{\mathrm{T}} = R_1(\bar{\imath})\, R_3(\bar{\Omega})\, R_3(-\Omega)\, R_1(-i)\, (r\cos u \quad r\sin u \quad 0)^{\mathrm{T}}. \tag{303}$$

We want to use (303) as source for perturbation formulae that are correct to second order. Now $R_3(\bar{\Omega})\, R_3(-\Omega) = R_3(\bar{\Omega} - \Omega)$ and, of course, $\Omega - \bar{\Omega} = \delta\Omega$ by the basic definition (43). We write δ in place of $\delta\Omega$, for convenience, and also write

$$\iota = i - \bar{\imath} \quad (= \delta i),$$

with the sine and cosine of i and $\bar{\imath}$ denoted by s, c, $\bar{\mathrm{s}}$ and $\bar{\mathrm{c}}$. Then the second-order approximation of (303) is

$$\begin{pmatrix} r'\cos u' \\ r'\sin u' \\ c \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{2}\delta^2 & -\mathrm{c}\delta & \mathrm{s}\delta \\ \bar{\mathrm{c}}\delta & 1 - \frac{1}{2}\iota^2 - \frac{1}{2}\mathrm{c}^2\delta^2 & -\iota + \frac{1}{2}\mathrm{sc}\delta^2 \\ -\bar{\mathrm{s}}\delta & \iota + \frac{1}{2}\mathrm{sc}\delta^2 & 1 - \frac{1}{2}\iota^2 - \frac{1}{2}\mathrm{s}^2\delta^2 \end{pmatrix} \begin{pmatrix} r\cos u \\ r\sin u \\ 0 \end{pmatrix}.$$

The individual equations may be written as

$$r'\cos u' = r(\cos u - \mathrm{c}\delta\sin u - \tfrac{1}{2}\delta^2\cos u), \tag{304}$$

$$r'\sin u' = r[\sin u + \bar{\mathrm{c}}\delta\cos u - \tfrac{1}{2}(\iota^2 + \mathrm{c}^2\delta^2)\sin u] \tag{305}$$

and

$$c = r(\iota\sin u - \bar{\mathrm{s}}\delta\cos u + \tfrac{1}{2}\mathrm{sc}\delta^2 \sin u). \tag{306}$$

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To first order, we again see that $r' = r$ and recover the results given by (173) and (174) of § 6.3.

To second order, we still have identity between r' and r, consequent on our choice of k_i, as remarked at the end of § 3.2. Thus we need not consider r' as a separate quantity, but note that we would have to do so if we were concerned with $O(K^2e^2)$ perturbations.

We can now obtain the second-order perturbation in u', using either (304) or (305) – as a check we use *both*. From (304) we have

$$\cos u' = \cos u - (\bar{c} - \bar{s}\iota)\,\delta \sin u - \tfrac{1}{2}\delta^2 \cos u, \tag{307}$$

where $\delta = \bar{K}\Omega_1 + \bar{K}^2\Omega_2$ and $\iota = \bar{K}i_1$, while from (305) we have

$$\sin u' = \sin u + \bar{c}\delta \cos u - \tfrac{1}{2}(\iota^2 + c^2\delta^2) \sin u. \tag{308}$$

But if v denotes $u' - u$, we have the Taylor expansions

$$\cos u' = \cos u - v \sin u - \tfrac{1}{2}v^2 \cos u$$

and

$$\sin u' = \sin u + v \cos u - \tfrac{1}{2}v^2 \sin u.$$

We identify the two expressions for $\cos u'$, and likewise for $\sin u'$, expanding v according to the formula

$$v = \bar{K}v_1 + \bar{K}^2 v_2, \tag{309}$$

where v_1 and v_2 are required. It is immediate that

$$v_1 = \bar{c}\Omega_1, \tag{310}$$

but we have two formulae for v_2, namely

$$(v_2 - \bar{c}\Omega_2) \sin u = -\bar{s}i_1\Omega_1 \sin u + \tfrac{1}{2}(\Omega_1^2 - v_1^2) \cos u \tag{311}$$

from the $\cos u'$ identity and

$$(v_2 - \bar{c}\Omega_2) \cos u = \tfrac{1}{2}(v_1^2 - i_1^2 - c^2\Omega_1^2) \sin u \tag{312}$$

from the $\sin u'$ identity. On substitution for v_1, i_1 and Ω_1, we get the same formula in both cases,

$$v_2 - \bar{c}\Omega_2 = -\tfrac{1}{16}f(1-f)\,(\sin 4\bar{u} + 2 \sin 2\bar{u}). \tag{313}$$

From (226), the formula for Ω_2, we now get

$$v_2 = -\tfrac{1}{48}(1-f)\,\{(6+5f) \sin 4\bar{u} + 2(24-39f) \sin 2\bar{u}\}. \tag{314}$$

For u_2', it remains to combine (302) and (314), the result being

$$u_2' = -\tfrac{1}{72}f\{f \sin 4\bar{u} - (19-20f) \sin 2\bar{u}\}. \tag{315}$$

We finally obtain c_1 and c_2, where

$$c(=\delta c) = \bar{K}c_1 + \bar{K}^2 c_2, \tag{316}$$

from (306). We consider this equation in the form

$$c/r = \bar{K}(i_1 \sin u - \bar{s}\Omega_1 \cos u) + \bar{K}^2(i_2 \sin u - \bar{s}\Omega_2 \cos u + \tfrac{1}{2}sc\Omega_1^2 \sin u). \tag{317}$$

Now from (149) and (150), identifying u and $\bar{u}$ only in the $\bar{e}$ term,

$$i_1 \sin u - \Omega_1 \sin \bar{\imath} \cos u = \tfrac{1}{2} \sin \bar{\imath} \cos \bar{\imath}\{[\sin u - \sin(2\bar{u} - u)] + \tfrac{4}{3}\bar{e}[2 \sin(\bar{u} + \bar{v}) - 3 \sin \bar{\omega}]\}. \tag{318}$$

Thus we have recovered (174), except that there is a 'second-order carry-over' given by the first term of (318), which can be written as $\tfrac{1}{2}(u - \bar{u}) \sin 2\bar{\imath} \cos \bar{u}$. The second-order terms now combine to give

$$c_2/r = i_2 \sin u - \Omega_2 \sin \bar{\imath} \cos u + \tfrac{1}{4}\Omega_1^2 \sin 2\bar{\imath} \sin u + \tfrac{1}{2}u_1 \sin 2\bar{\imath} \cos u, \tag{319}$$

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which from (217), (226), (150), (53) and (173), if we replace u by $\bar{u}$, reduces to

$$c_2 = -\tfrac{1}{96} r \sin 2\bar{\imath}\{8\bar{f} \sin 3\bar{u} - (33 - 49\bar{f} - k_{2i}) \sin \bar{u}\}. \tag{320}$$

The reason for choosing k_{2i} according to (218) should now be apparent, and the final result is simply

$$c_2 = -\tfrac{1}{12} r\bar{f} \sin 2\bar{\imath} \sin 3\bar{u}. \tag{321}$$

9. Complete secular and long-periodic perturbations due to J_2^2

9.1. *General remarks*

The original intention was to limit the analysis of J_2^2-perturbations in this paper strictly to e-independent terms, and the resulting expressions have been seen to be very simple. However, for an eccentricity of 0.01 say, a low-altitude satellite experiences long-term perturbations of order $J_2^2 e$ that, within about a day (since this corresponds to an angular motion of about 100 radians), are of the same order of magnitude as the short-period perturbations of order J_2^2. Hence the paper would not be complete without consideration of these long-term effects. Once we go to order $J_2^2 e$, there is not much difficulty in giving the full formulae, valid for any eccentricity. They are quoted without full proof, since derivations are available in Merson (1966) – Merson, like Gooding (1966), gives complete first-order formulae for the long-term perturbations due to the general zonal harmonic, J_l.

The long-term e-independent perturbations are purely secular. When e-dependent terms are considered, however, the long-term perturbations contain long-periodic components, trigonometrically related to the argument of perigee (ϖ), as well as purely secular components. The nature of these perturbations is frequently misunderstood, so some general remarks are offered before the results for the elements a, e, i, Ω, ω and M are given. (There is no need to go to non-singular elements ξ, η and U, since e^{-1} factors do not appear in the long-term perturbations.)

Suppose a term $K\bar{n} \cos \mathrm{k}\varpi$ (with K not necessarily the J_2-related quantity used in this paper) occurs as a component of the rate of change of some element, ζ say. If ϖ is constant (as it is for orbits at the critical inclination) we can integrate to get a secular perturbation $K\bar{n}t \cos \mathrm{k}\varpi$. If ϖ is not constant, but is itself essentially secular with rate $\dot{\varpi}$, the perturbation can be written (in the Δ notation of § 3.1) in the form

$$\Delta\zeta = (K\bar{n}/\mathrm{k}\dot{\varpi}) [\![\sin \mathrm{k}\varpi]\!],$$

where $[\![\ldots]\!]$ designates 'variation over the period of integration', i.e. it is a definite integral. The two results are compatible, since $[\![\sin \mathrm{k}\varpi]\!]/\mathrm{k}\dot{\varpi}$ tends to $t \cos \mathrm{k}\varpi$, over the interval t, as $\dot{\varpi}$ tends to zero. If, however, $\Delta\zeta$ is written with an arbitrary lower limit of integration, as with short-periodic perturbations, in particular in the form $\sin \mathrm{k}\varpi/\mathrm{k}\dot{\varpi}$, the perturbation has an apparent singularity for $\dot{\varpi}$ zero.

As was made clear by Merson (1966), trouble is avoided if we keep to definite integrals. Since we are concerned with $\mathrm{k} = 2$, we denote by I_c and I_s the time integrals of $\cos 2\varpi$ and $\sin 2\varpi$ – Merson denoted the negatives of these quantities by C_2 and S_2 respectively and showed how they might be neatly evaluated.

The difficulty is largely a matter of terminology†. The terms 'first-order', 'second-order', etc.,

† The long-standing division of opinion over the problem of the critical inclination is a good example of the semantic confusion related to the meaning of 'order'. As Allan (1970) has pointed out, the behaviour of certain of the orbital elements over very long periods of time is indeed a libration of amplitude proportional to $J_2^{\frac{1}{2}}$, but this does not imply a singularity – the effect arises from *rates of change* that are still only of order J_2^2.

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are uncontroversial when used to describe short-periodic and secular perturbations, a second-order (J_2^2) perturbation having the intuitive meaning that the perturbation would have been λ^2 times smaller if J_2 had been λ times smaller. It is not so simple for a long-periodic perturbation, however, since the frequency of such a perturbation itself varies (linearly) with J_2 and there is no longer such an obvious meaning; a long-periodic component of $\dot{\zeta}$ proportional to J_2^2 leads to a $\Delta\zeta$ term whose amplitude falls only in proportion to λ when the $\dot{\zeta}$ component falls in proportion to λ^2, but it is misleading to interpret this by deeming the $\Delta\zeta$ term to become first-order in J_2.

There is no problem if we take the order of a long-periodic perturbation from the power of J_2 appearing in the appropriate rate-of-change component, i.e. in $\dot{\zeta}$ for element ζ. With this philosophy, there are no long-periodic perturbations of first-order (though J_2 is exceptional here and there *are* first-order perturbations in J_3, etc.). Second-order perturbations arise, and may be expressed in terms of $\bar{K}^2\bar{n}I_c$ and $\bar{K}^2\bar{n}I_s$. Since

$$I_c = [\![\sin 2\bar{\omega}]\!]/2\dot{\bar{\omega}},$$

where

$$\dot{\bar{\omega}} = \tfrac{1}{2}\bar{K}\bar{n}(4-5\bar{f}),$$

we have

$$\bar{K}^2\bar{n}I_c = \bar{K}[\![\sin 2\bar{\omega}]\!]/(4-5\bar{f})$$

(and $\bar{K}^2\bar{n}I_s$ similarly). This can be verified as still of *second* order, according to the theory of asymptotic development (Dieudonné 1968), even though it is superficially of first order and has become regularly so regarded in the literature. There appears to be no merit whatever in misrepresenting these perturbations since they are, after all, not merely less important than *first-order* secular perturbations, but are also less important than *second-order* secular perturbations. (Their long-term magnitude is admittedly much greater than for *short*-periodic perturbations of the second order, but this is no more than the usual more-important-if-longer-period effect.)

This is not the full story, however, as we encounter some apparently first-order long-periodic perturbations that are really of *third* order. The dominant secular perturbations $\dot{\bar{\Omega}}$ and $\dot{\bar{\omega}}$, given by $-\bar{K}\bar{n}\cos\bar{\imath}$ and $\frac{1}{2}\bar{K}\bar{n}(4-5\bar{f})$ respectively, are functions of $\bar{e}$ (an argument of $\bar{K}$) and $\bar{\imath}$, so that they have induced long-periodic variation – thus $\cos\bar{\imath}$ is equal to an initial (hence constant) value, $\cos\bar{\imath}_0$, plus a term given by $-\Delta i\sin\bar{\imath}_0$. The integral of $\bar{K}\Delta i$ is itself long-periodic and of third order, but because two integrations are now involved it is apparently first-order, particularly if written with a factor $\dot{\bar{\omega}}^{-2}$. Since, as we shall see, Δe and Δi are multiples of I_s, it is natural to introduce the notation II_s for the integral of this quantity, i.e. for the double time integral of $\sin 2\bar{\omega}$.

Two further introductory points must be made. First, the form each long-periodic perturbation takes is dependent on the expression for the first-order short-periodic perturbation, in that the arbitrary k-constant may be chosen, in different ways, as a function of $\bar{\omega}$. Secondly, there is an entirely different way of looking at second-order long-term variation, and this was the approach originally adopted by Merson (1961) and Gooding (1966), in particular. It involves the derivation of each element's variation over a complete revolution of the satellite, to eliminate the short-periodic effect, but now the interpretation of 'complete revolution' is crucial: the variation from perigee to perigee, for example, is different from the variation from node to node, not because perigee and node are different points but simply because there is a slippage (due to $\dot{\bar{\omega}}$), as between successive perigees and successive nodes.

9.2. *Perturbation in a*

There is no secular perturbation in the semi-major axis, nor is there any second-order long-periodic variation so long as the $\bar{\omega}$-dependent part of the 'constant' k_a is specifically $\frac{3}{4}\bar{f}\bar{e}^2\cos 2\bar{\omega}$.

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This is the case for all four of the k_a given in § 6.3, by (160), (179), (183) and (185). Given an arbitrary k_a, however, we have a long-periodic perturbation given by

$$\Delta a = \bar{K}\bar{a}\bar{q}^{-2}[\![\tfrac{3}{4}\bar{f}\bar{e}^2\cos 2\bar{\omega} - k_a]\!]. \tag{322}$$

As indicated at the end of § 9.1, there is an apparent long-periodic effect if we only consider the semi-major axis at, say, perigees or ascending nodes. For perigees, the effect is given by

$$\Delta a = -\bar{K}^2\bar{n}\,\bar{a}\,\bar{q}^{-2}\,(1+\bar{e})^3\bar{f}\,(4-5\bar{f})\,I_s,$$

as follows from equation (181) of Gooding (1966).

9.3. *Perturbation in e*

There is again no secular perturbation, but there is a long-periodic perturbation given by

$$\Delta e = -\tfrac{1}{24}\bar{K}^2\bar{n}\bar{e}\bar{q}^2\bar{f}(14-15\bar{f})\,I_s + \tfrac{1}{8}\bar{K}\bar{e}[\![3\bar{f}\cos 2\bar{\omega} - k_e]\!], \tag{323}$$

where the second term vanishes for both our standard k_e, given by (161), and the value implied by use of Kozai elements, as given by (192). Berger & Walch (1977) and Kinoshita (1977) implicitly use the value given by (194), whence it follows that, to $O(\bar{e})$, their expression for Δe should involve a factor $2(13-15\bar{f})$ in place of the factor $14-15\bar{f}$ in the standard expression. This is confirmed by line 15 of page 112 of Berger & Walch (1977).

9.4. *Perturbation in i*

A simple relation exists between perturbations in osculating a, e and i, since $p\cos^2 i$ is constant for any potential field that is symmetric about the z-axis, in particular for the potential associated with J_2. It follows that, for variations in the osculating elements,

$$\delta i = \cot i\,(\tfrac{1}{2}\delta a/a - e\delta e/q^2) \tag{324}$$

and the corresponding relation expressing Δi in terms of Δa and Δe also holds, so long as

$$4[\![k_a]\!] - \bar{e}^2[\![k_e]\!] - 4\bar{f}\bar{q}^2[\![k_i]\!] = 0,$$

which is satisfied both by our standard k's and the k's corresponding to unbiased mean elements. It follows that, for general k_i,

$$\Delta i = \tfrac{1}{48}\bar{K}^2\bar{n}\bar{e}^2\sin 2\bar{\imath}\,(14-15\bar{f})\,I_s - \tfrac{1}{4}\bar{K}\sin 2\bar{\imath}[\![k_i]\!], \tag{325}$$

the second term vanishing for both our standard k_i (unity) and the Kozai k_i (zero). However, Berger & Walch (1977) and Kinoshita (1977) implicitly use the value of k_i given by (195), whence it again follows, to $O(\bar{e})$, that their expression for Δi should involve a factor $2(13-15\bar{f})$ in place of the factor $14-15\bar{f}$ in the standard expression, and again this is confirmed.

9.5. *Perturbation in Ω*

The complete second-order long-term perturbation may be obtained from the following expression, effectively obtained by Merson (1966) for the rate of change of the mean element:

$$\frac{\mathrm{d}\bar{\Omega}}{\mathrm{d}t} = -\bar{K}\bar{n}\cos\bar{\imath}\Big[1+\frac{1}{2}\Big(\frac{\dot{\bar{\omega}}}{\bar{n}}\Big)\frac{\partial k_\Omega}{\partial\omega}+\tfrac{1}{24}\bar{K}\bar{q}^{-2}\{4(15-19\bar{f})+\bar{e}^2(4-9\bar{f})-\bar{e}^4(4+5\bar{f})$$
$$+\bar{e}^2\cos 2\bar{\omega}[(14-3\bar{f})-2\bar{e}^2(7-15\bar{f})]-48k_a+12\bar{e}^2k_e-12\bar{f}\bar{q}^2k_i-36k_n\}\Big]. \tag{326}$$

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We consider the secular component first. With our standard values of k_a, k_e and k_i, and with k_n given by (180), we get

$$\dot{\bar{\Omega}}_2 = \tfrac{1}{24}\bar{n}\cos\bar{\imath}\{4(3-5\bar{f})-\bar{e}^2(4+5\bar{f})\}, \tag{327}$$

a truncated form of which is given by (225) with $k_n = \frac{2}{3}\bar{h}$. With the k's of Kozai, on the other hand, we get

$$\dot{\bar{\Omega}}_2 = \tfrac{1}{24}\bar{n}\cos\bar{\imath}\{48\bar{h}\bar{q}-4(9-10\bar{f})-\bar{e}^2(4+5\bar{f})\}, \tag{328}$$

as given (effectively) by Kozai (1959) and Merson (1966). Finally, with the k's of Berger & Walch (1977) and Kinoshita (1977), we get

$$\dot{\bar{\Omega}}_2 = -\tfrac{1}{24}\bar{n}\cos\bar{\imath}\{24\bar{h}\bar{q}+4(9-10\bar{f})+\bar{e}^2(4+5\bar{f})\}, \tag{329}$$

in agreement with results in these papers – since Berger & Walch use $(R/\bar{a})^4$ rather than $(R/\bar{p})^4$ as (effectively) a factor of $\bar{K}$, the bracketed expression here must be multiplied by $\bar{q}^{-8}$ before agreement is evident.

Turning to the long-periodic component, we can write the basic second-order perturbation $\Delta\Omega_b$, as

$$\begin{aligned}\Delta\Omega_b = &-\tfrac{1}{12}\bar{K}^2\bar{n}\bar{e}^2(7-15\bar{f})\cos\bar{\imath}\, I_c - \tfrac{1}{2}\bar{K}\cos\bar{\imath}\,[\![k_\Omega]\!]\\ &+\tfrac{1}{8}\bar{K}^2\bar{n}\bar{q}^{-2}\cos\bar{\imath}\int\{4[4(k_a)_\omega-3\bar{f}\bar{e}^2\cos 2\bar{\omega}]\\ &-4\bar{e}^2[(k_e)_\omega-3\bar{f}\cos 2\bar{\omega}]+4\bar{f}\bar{q}^2(k_i)_\omega+3[4(k_n)_\omega-3\bar{f}\bar{e}^2\cos 2\bar{\omega}]\}\,dt,\end{aligned} \tag{330}$$

where $(k_a)_\omega$ denotes the ω-dependent (long-periodic) part of k_a, etc. But in addition we now have an induced (third-order) perturbation $\Delta\Omega_{ind}$, as explained in § 9.1. On the assumption that the first-order secular perturbation is computed from

$$\dot{\bar{\Omega}} = -\bar{K}_0\bar{n}_0\cos\bar{\imath}_0,$$

where zero-suffices have been added to indicate that we use *initial* values of the mean elements, our additional perturbation is given by

$$\Delta\Omega_{ind} = -\bar{K}\bar{n}\cos\bar{\imath}\left(4\bar{e}\bar{q}^{-2}\int\Delta e\,dt-\tan\bar{\imath}\int\Delta i\,dt\right); \tag{331}$$

from this, using (324), we may write

$$\Delta\Omega_{ind} = -5\bar{K}\bar{n}\bar{e}\bar{q}^{-2}\cos\bar{\imath}\int\Delta e\,dt. \tag{332}$$

Hence our standard expression for the combination of $\Delta\Omega_b$ and $\Delta\Omega_{ind}$ is given by

$$\Delta\Omega = -\tfrac{1}{24}\bar{K}^2\bar{n}\bar{e}^2\cos\bar{\imath}\{2(7-15\bar{f})I_c-5\bar{K}\bar{n}\bar{f}(14-15\bar{f})II_s\}. \tag{333}$$

Use of the k's of Berger & Walch (1977) and Kinoshita (1977), on the other hand, including k_Ω by (196), leads to

$$\Delta\Omega = -\tfrac{1}{24}\bar{K}^2\bar{n}\bar{e}^2\cos\bar{\imath}\{[2(13-30\bar{f})+O(\bar{e}^2)]I_c-10\bar{K}\bar{n}\bar{f}(13-15\bar{f})II_s\}; \tag{334}$$

if (by taking indefinite integrals instead of evaluating I_c and II_s – see the discussion in § 9.1), we interpret this as a first-order perturbation, we get

$$-\frac{\bar{K}\bar{e}^2\cos\bar{\imath}(52-120\bar{f}+75\bar{f}^2)}{12(4-5\bar{f})^2}\sin 2\bar{\omega}+O(\bar{e}^4), \tag{335}$$

in agreement with line 20 of page 112 of Berger & Walch (1977).

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9.6. *Perturbation in ω*

The complete second-order long-term perturbation may be obtained from the following expression, effectively obtained by Merson (1966), for the rate of change of the mean element:

$$\frac{\mathrm{d}\bar\varpi}{\mathrm{d}t} = \tfrac{1}{2}\bar K\bar n\Big[(4-5\bar f) - \frac{1}{4}\left(\frac{\dot{\bar\omega}}{\bar n}\right)\frac{\partial k_\omega}{\partial\omega} + \tfrac{1}{48}\bar K\bar q^{-2}\{2(288-676\bar f+395\bar f^2) - \bar e^2(40+4\bar f-65\bar f^2)$$
$$-\bar e^4(56-36\bar f-45\bar f^2) - 2\cos 2\bar\varpi[20(5-6\bar f) - \bar e^2(28-4\bar f+15\bar f^2) + \bar e^4(28-158\bar f+135\bar f^2)]$$
$$-24(4-5\bar f)(4k_a-\bar e^2k_e+3k_n) - 240\bar f(1-\bar f)\bar q^2k_i\}\Big]. \tag{336}$$

We consider the secular component first. With our standard values of k_a, k_e and k_i, and with k_n given by (180), we get

$$\dot{\bar\omega}_2 = -\tfrac{1}{96}\bar n\{2\bar f(4+25\bar f) - \bar e^2(56-36\bar f-45\bar f^2)\}. \tag{337}$$

With the k's of Kozai, on the other hand, we get

$$\dot{\bar\omega}_2 = -\tfrac{1}{96}\bar n\{96\bar h\bar q(4-5\bar f^2) - 2(192-412\bar f+215\bar f^2) - \bar e^2(56-36\bar f-45\bar f^2)\}, \tag{338}$$

as given (effectively) by Kozai (1959) and Merson (1966). Finally, with the k's of Berger & Walch (1977) and Kinoshita (1977) we get

$$\dot{\bar\omega}_2 = \tfrac{1}{96}\bar n\{48\bar h\bar q(4-5\bar f) + 2(192-412\bar f+215\bar f^2) + \bar e^2(56-36\bar f-45\bar f^2)\}, \tag{339}$$

in agreement with results in these papers; as with $\dot{\bar\Omega}_2$, the bracketed expression must be multiplied by $\bar q^{-8}$ to make agreement with Berger & Walch evident.

Turning to the long-periodic component, we can write the basic second-order perturbation as

$$\Delta\omega_{\mathrm{b}} = -\tfrac{1}{48}\bar K^2\bar n\{2\bar f(14-15\bar f) - \bar e^2(28-158\bar f+135\bar f^2)\}I_{\mathrm{c}} - \tfrac{1}{8}\bar K[\![k_\omega + 3\bar f\sin 2\bar\varpi]\!]$$
$$-\tfrac{1}{16}\bar K^2\bar n\bar q^{-2}\int\langle(4-5\bar f)\{4[4(k_a)_\omega - 3\bar f\bar e^2\cos 2\bar\varpi] - 4\bar e^2[(k_e)_\omega - 3\bar f\cos 2\bar\varpi]$$
$$+3[4(k_n)_\omega - 3\bar f\bar e^2\cos 2\bar\varpi]\} + 40\bar f(1-\bar f)\bar q^2(k_i)_\omega\rangle\,\mathrm{d}t, \tag{340}$$

with the same notation as for $\Delta\Omega_{\mathrm{b}}$. We also have an induced perturbation $\Delta\omega_{\mathrm{ind}}$, that arises from computation of the first-order secular perturbation by using

$$\dot{\bar\omega} = \tfrac{1}{2}\bar K_0\bar n_0(4-5\bar f_0).$$

This additional perturbation is given by

$$\Delta\omega_{\mathrm{ind}} = \bar K\bar n\left[2\bar e\bar q^{-2}(4-5\bar f)\int\Delta e\,\mathrm{d}t - 5\sin\bar\imath\cos\bar\imath\int\Delta i\,\mathrm{d}t\right]; \tag{341}$$

this gives, by using (324),

$$\Delta\omega = \bar K\bar n\bar e\bar q^{-2}(13-15\bar f)\int\Delta e\,\mathrm{d}t. \tag{342}$$

Hence our standard expression for the combination of $\Delta\omega_{\mathrm{b}}$ and $\Delta\omega_{\mathrm{ind}}$ is given by

$$\Delta\omega = -\tfrac{1}{48}\bar K^2\bar n\{[2\bar f(14-15\bar f) - \bar e^2(28-158\bar f+135\bar f^2)]I_{\mathrm{c}} + 2\bar K\bar n\bar e^2\bar f(13-15\bar f)(14-15\bar f)II_{\mathrm{s}}\}. \tag{343}$$

Use of the k's of Berger & Walch (1977) and Kinoshita (1977), on the other hand, including k_ω by (197), leads to

$$\Delta\omega = -\tfrac{1}{24}\bar K^2\bar n\{[2\bar f(13-15\bar f) - \bar e^2(26-155\bar f+140\bar f^2) + O(\bar e^4)]I_{\mathrm{c}} + 2\bar K\bar n\bar e^2\bar f(13-15\bar f)^2II_{\mathrm{s}}\}; \tag{344}$$

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if we take the apparent first-order perturbation that results, we get

$$-\frac{\bar{K}\sin 2\bar{\omega}}{24(4-5\bar{f})^2}\{2\bar{f}(52-125\bar{f}+75\bar{f}^2)-\bar{e}^2(104-412\bar{f}+555\bar{f}^2-250\bar{f}^3)+O(\bar{e}^4)\} \qquad (345)$$

and this conforms with lines 23 and 24 of page 112 of Berger & Walch (1977) if we remember to replace $\bar{K}$ by $\bar{K}\bar{q}^4(1-\bar{e}^2)^{-2}$, so that the coefficient of $\bar{e}^2$ changes to

$$-(104-620\bar{f}+1055\bar{f}^2-550\bar{f}^3).$$

9.7. *Perturbation in M*

Merson (1966) has effectively shown that the rate of change of $\bar{M}$ is given by

$$\mathrm{d}M/\mathrm{d}t = n' - \tfrac{1}{48}\bar{K}^2\bar{n}\bar{q}^3\{(8-8\bar{f}-5\bar{f}^2)-2\bar{f}(14-15\bar{f})\cos 2\bar{\omega}\}+\tfrac{1}{8}\bar{K}\bar{q}\dot{\bar{\omega}}(\partial k_M/\partial\omega+6\bar{f}\cos 2\bar{\omega}), \quad (346)$$

where the final term has been added here to cover the possibility of a non-standard value of k_M. It is noteworthy, in this basic formula, that there is only a second-order term, residual to the absolute constant n' introduced in § 3.

Equation (346) could be generalized to replace n' by an arbitrary $\bar{n}$, with explicit appearance of arbitrary k_n and k_{2n} (it would be necessary to write $\bar{n}_0$, to emphasize *initial* value, if the $\bar{f}$-component were other than $\frac{3}{4}\bar{f}\bar{e}^2\cos 2\bar{\omega}$) but little is to be gained by doing this. In fact it makes no sense to try to give an accurate (untruncated) general expression for the overall secular variation, $\dot{\bar{M}}$, since components that are $O(\bar{K}^2\bar{e})$ can always be omitted on the basis of their equivalence to constant $O(\bar{K}^2\bar{e})$ components in the semi-major axis, i.e. $O(\bar{e})$ components of k_{2a}, which we continue to neglect. Further, replacing n' would have no effect on the form of the long-periodic perturbation, ΔM, which depends only on the choice of k_M.

However, two things are worth doing in connection with the secular variation. First, it is desirable to identify the $\bar{n}$ which accounts for $\dot{\bar{M}}$ completely, i.e. to evaluate k_{2n} (necessarily truncated, for the reason given in the last paragraph), and hence $\hat{\mu}_2$ (as defined in § 7.8) in the expression required for Kepler's third law, such that $\dot{\bar{M}}_2$ (rather than $\dot{\bar{U}}_2$, which was the aim in § 7.9) vanishes. Secondly, it is of interest to replace n' by the unbiased $\bar{n}$ of Berger & Walch (1977), by working only to $O(\bar{K}^2\bar{e})$ again (for the same reason); the reward for this exercise is an overall check between Merson (1966), Berger & Walch (1977) and the particular formula, (204), for k_{2a} in § 7.1. (The starting point for an untruncated check would be a derivation of the complete second-order perturbation in osculating semi-major axis, the simplest such derivation being by the 'special method' of § 7.1.)

To make $\dot{\bar{M}}_2$ vanish, without going through the process of generalizing (346), we go back to $\dot{\bar{U}}_2$, as given by (274), and $\dot{\bar{\omega}}$, which can be derived from (336). Then subtraction gives

$$\dot{\bar{M}}_2 = \tfrac{1}{24}\bar{n}\{(72-188\bar{f}+121\bar{f}^2)-6(2-3\bar{f})(4k_a+3k_n)-36k_i\bar{f}(1-\bar{f})+24k_{2n}\}. \qquad (347)$$

Taking our standard values of k_a and k_i, and with k_n truncated from (180), we get

$$\dot{\bar{M}}_2 = -\tfrac{1}{24}\bar{n}\{\bar{f}(8+5\bar{f})-24k_{2n}\}. \qquad (348)$$

For this to vanish, clearly

$$k_{2n} = \tfrac{1}{24}\bar{f}(8+5\bar{f}). \qquad (349)$$

Using (349), we can easily derive the expression for $\hat{\mu}_2$ required to extend the first-order modified version of Kepler's third law, in preferred form (for untruncated analysis) as given by

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(182), to second order. We start with the general formula for $\hat{\mu}_2$ given by (261). Substituting $\bar{h}$ for k_a and $\frac{2}{3}\bar{h}$ for k_n, we get

$$\hat{\mu}_2 = \tfrac{3}{8}(5\bar{f}^2 + 8\bar{h}^2) - 3k_{2a} - 2k_{2n}. \quad (350)$$

Then with the preferred k_{2a}, given by (211), and the expression for k_{2n} just derived, i.e. (349), we get

$$\hat{\mu}_2 = -\tfrac{1}{6}\bar{f}(4 - 9\bar{f}). \quad (351)$$

This is a truncated result, but it is effectively untruncated in view of the equivalence of an $O(\bar{e})$ component of $\hat{\mu}_2$ to an $O(\bar{K}^2\bar{e})$ constant term in $\bar{a}$ that has been remarked.

We now undertake the exercise of replacing n' in (346) by the unbiased $\bar{n}$ of Berger & Walch (1977), to arrive at the indicated 'overall check' from the value of $\dot{\bar{M}}_2$ that results. Now the values of k_{2n} appropriate to n' and $\bar{n}$ (the $\bar{n}$ of Berger & Walch will be understood in what follows) can be derived from the values of k_a and k_{2a} associated with the corresponding a' and $\bar{a}$, respectively, on taking $\hat{\mu}_1$ and $\hat{\mu}_2$ as zero in both cases. Thus (254) gives $k_n = k_a$ and then (261) gives

$$k_{2n} = \tfrac{3}{16}(5\bar{f}^2 + 10k_a^2 - 8k_{2a}). \quad (352)$$

For n' (such that $n'^2a'^3 = \mu$) we have k_a given by (179) and k_{2a} given, to $O(\bar{e})$, by (204). This leads to what we may designate $k_{2n}(n')$, an expression for which has already been given – it is (281) with a general k_i, or (282) when $k_i = 1$. For $\bar{n}$ (which satisfies $\bar{n}^2\bar{a}^3 = \mu$), on the other hand, both k_a and k_{2a} are $O(\bar{e}^2)$ so that, to $O(\bar{e})$, $k_{2n}(\bar{n})$ is $\frac{15}{16}\bar{f}^2$.

But the two values of k_{2n} are defined such that

$$n'\{1 - \tfrac{3}{2}K'k_n(n') + K'^2k_{2n}(n')\}$$

and

$$\bar{n}\{1 - \tfrac{3}{2}\bar{K}k_n(\bar{n}) + \bar{K}^2k_{2n}(\bar{n})\}$$

are identical, to $O(\bar{e})$, since they represent the same mean value of the osculating mean motion, n. We make the identification, taking k_i to be 0, not 1, in (281), since it is $O(\bar{e}^2)$ in (195). Also, $k_n(n') = \frac{2}{3}\bar{h}$, to $O(\bar{e})$, and $k_n(\bar{n}) = 0$, to $O(\bar{e})$, so

$$n' = \bar{n} + K'n'\bar{h} + \tfrac{1}{48}\bar{K}^2\bar{n}(40 - 48\bar{f} + 30\bar{f}^2). \quad (353)$$

We must replace $K'n'$ by $\bar{K}\bar{n}$ in the first-order term of this relation, but in fact

$$K'n' = \bar{K}\bar{n}(1 + \tfrac{7}{3}\bar{K}\bar{h}) + O(\bar{K}^2\bar{e}^2),$$

since, to $O(\bar{e})$, $K' = \bar{K}(1 + \frac{4}{3}\bar{K}\bar{h})$ and $n' = \bar{n}(1 + \bar{K}\bar{h})$. Thus we finally get

$$n' = \bar{n} + \bar{K}\bar{n}\bar{h} + \tfrac{1}{48}\bar{K}^2\bar{n}(152 - 384\bar{f} + 282\bar{f}^2) + O(\bar{e}^2). \quad (354)$$

On combining the $\bar{K}^2$ terms in (346) and (354), we see that, residual to $\bar{n}$ (unbiased) and the first-order term $\bar{K}\bar{n}\bar{h}$, $\mathrm{d}\bar{M}/\mathrm{d}t$ contains the secular component

$$\tfrac{1}{48}\bar{K}^2\bar{n}(144 - 376\bar{f} + 287\bar{f}^2),$$

and this is the result that was required, since it is in agreement with line 30 of page 110 of Berger & Walch (1977). It does not tally with the fourth formula of page B-53 of Kinoshita (1977), but it has already been remarked (in § 7.10) that an apparent error in Kinoshita's second-order $\dot{\bar{M}}$ can be resolved by (effective) allowance for a non-zero k_{2a}.

We now return to (346), for the long-periodic perturbation in $\bar{M}$. It follows at once that the general expression for ΔM is given by

$$\Delta M = \tfrac{1}{24}\bar{K}^2\bar{n}\bar{q}^3\bar{f}(14 - 15\bar{f})\, I_c + \tfrac{1}{8}\bar{K}\bar{q}[\![k_M + 3\bar{f}\sin 2\bar{\omega}]\!]. \quad (355)$$

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From (162), our 'standard expression' is given by the first term of (355). To obtain the appropriate expression to check against Berger & Walch (1977), however, we require the k_M given by (198), and this leads to

$$\Delta M = \tfrac{1}{384}\bar{K}^2\bar{n}\bar{f}\{32(13-15\bar{f})-80\bar{e}^2(7-8\bar{f})+3\bar{e}^4(32-35\bar{f})+O(\bar{e}^6)\}I_c. \tag{356}$$

Expressed as an apparent first-order perturbation, as usual, and with $\bar{K}$ then replaced by $\bar{K}\bar{q}^4(1-\bar{e}^2)^{-2}$, (356) gives

$$\frac{\bar{K}\bar{q}^4\bar{f}\sin 2\bar{\omega}}{384(4-5\bar{f})}\{32(13-15\bar{f})+16\bar{e}^2(17-20\bar{f})+\bar{e}^4(224-265\bar{f})+O(\bar{e}^6)\},$$

and this conforms with lines 27 and 28 of page 112 of Berger & Walch (1977), if it is observed that lines 27–30 of their paper all contain an unnecessary factor, $4-5\bar{f}$, that may be divided out.

10. Concluding remarks

The introduction to this paper referred to the difficulty, in much of the existing literature, in picking out the main results from a mass of mathematics. To avoid the same difficulty here, therefore, a summary will be given of the main results, referenced by equation number.

Equations (84)–(88) and (91) express the first-order e-independent perturbations in orbital elements due to the potential U^{k}_{nm} (expressed by (72)) associated with harmonic coefficients C_{nm} and S_{nm} through equation (33). The perturbations specified by (88) and (91), namely $\delta\psi$ and δL, are related to perturbations in standard elements, ω and M, by (8) and (9).

Correspondingly general expressions for perturbations in a system of cylindrical coordinates, based on the mean orbital plane, take the compact form indicated by (92)–(94). Combined such perturbations, namely $\delta r'$, $\delta u'$ and δc, are incorporated in the algorithm for satellite position (x, y, z) from mean elements at epoch as follows: secular and long-periodic perturbations are applied to each mean element, after (44); $\bar{r}$ and $\bar{u}$ are obtained from (45)–(48) and (18), at which point incorporation of $\delta r'$ and $\delta u'$ gives r' and u', c being simply δc; then (54) gives x, y and z. (Note: general expressions for long-periodic perturbations were given by Gooding (1966) and Merson (1966), and are omitted from the present paper.)

Untruncated first-order secular perturbations associated with the particular zonal harmonic J_2 are given by (126)–(132), with K, f, h and q given by (124), (14), (15) and (17), respectively, and short-periodic perturbations by (138)–(146). The untruncated expressions for perturbations in the cylindrical coordinates are (with the factor $\bar{K}$ omitted, after the notation of (147)) given by (163), (168) and (171); the corresponding truncated expressions (to cover $J_2 e$ terms but not $J_2 e^2$ terms) are (172)–(174). The compactness of these expressions (in comparison with evaluation of the general (92)–(94)) is due to a preferred choice of values for the arbitrary k-constants that appear in (138)–(146), namely the choice given by (160), (161), (170), (166), (167) and (162) for a, e, i, Ω, ω and M respectively. The k's that implicitly apply to other sets of mean orbital elements in common use are as follows: for Kozai's elements, k_a, k_e and k_i are given by (183), (192) and (193), the others being as 'preferred'; for Brouwer's elements, k_a is given by (179), the others being as for Kozai's elements; for elements with unbiased means over time, the k's are given by (18ɔ) and (194)–(198).

A seventh k, k_n, is used which must be equal to k_a if the Kepler relation (4) is to be preserved. To identify $\bar{n}$ with $\dot{\bar{M}}$, however, requires k_n to be given by (180), in which case (assuming the

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preferred k_a) the Kepler relation becomes (182). When working to $O(\bar{K}\bar{e})$ only (N.B. if $p \gg R$, Ke may be negligible even if $J_2 e$ is not), it is more appropriate to identify $\bar{n}$ with $\dot{\bar{U}}$ rather than $\dot{\bar{M}}$, with U given by (3), in which case k_n is given by (189) and the Kepler relation by (190). It is sometimes convenient, as indicated by Gooding (1974), to drop $\dot{\bar{\omega}}$ when working only to $O(\bar{K}\bar{e})$; this is possible, so long as terms $\frac{1}{2}\bar{a}\,\bar{e}\bar{n}t\,(4-5\bar{f})\sin\bar{v}$ and $\bar{e}\,\bar{n}t(4-5\bar{f})\cos\bar{v}$ are subtracted from (172) and (173) respectively.

Turning to second-order perturbations due to J_2, we assume first that we want untruncated accuracy in the long-term variation of mean elements, though only e-independent terms in the short-periodic perturbations given by (210), (217), (226), (233), (239), (250) and (273) – this amounts to the assumption that $\bar{K}^3\bar{n}t$ and $\bar{K}^2\bar{e}$ are negligible. Then mean elements at time t are given, from mean elements at epoch, by the combination of the secular perturbations in Ω and ω, specified by (327) and (337) respectively, with the long-periodic perturbations given by the first terms of (323) and (325), for e and i respectively, and (333), (343) and the first term of (355) for Ω, ω and M respectively. The absence of a second-order secular perturbation in M requires, to $O(\bar{K}^2\bar{e})$, that (after (182) and (351)) the Kepler relation between $\bar{n}$ and $\bar{a}$ be

$$\bar{n}^2\bar{a}^3 = \mu\{1-\bar{K}_0\bar{h}_0\bar{q}_0^{-2}(1-3\bar{e}_0^2)-\tfrac{1}{6}\bar{K}_0^2\bar{f}_0(4-9\bar{f}_0)\}, \tag{357}$$

wherein zero-suffices imply evaluation at epoch. The compact expressions for the perturbations in cylindrical coordinates, to be combined with (163), (168) and (171), are (297), (315) and (321).

Over short periods of time, such that $\bar{K}^2\bar{e}\bar{n}t$ can be neglected as well as $\bar{K}^2\bar{e}$, the second-order secular term in $\bar{\Omega}$ can be truncated to $\frac{1}{6}\bar{K}^2\bar{n}\cos\bar{\imath}(3-5\bar{f})\,t$, and the second-order secular term in $\bar{\omega}$ and all long-periodic terms dropped altogether. Dropping $\dot{\bar{\omega}}_2$ is dependent on a change in the Kepler relation, of which the second-order component is derived from the vanishing of (274) with k_n given by $\frac{2}{3}\bar{h}$ (since $\dot{\bar{M}}_1$, not $\dot{\bar{U}}_1$ as in §7.9, has to be zero); the relation becomes

$$\bar{n}^2\bar{a}^3 = \mu\{1-\bar{K}\bar{h}\bar{q}^{-2}(1-3\bar{e}^2)-\tfrac{1}{24}\bar{K}^2\bar{f}(20-11\bar{f})\}, \tag{358}$$

with zero-suffices no longer needed since $\bar{e}$ and $\bar{\imath}$ are assumed constant.

Finally, if $\bar{e}^2 < \bar{K}$ it may be possible to truncate first-order short-periodic expressions, $\bar{K}\bar{e}^2$ (but not $\bar{K}\bar{e}$ or $\bar{K}^2$) being negligible. Then we can truncate the $\bar{K}$-term in (358) to simply $\bar{K}\bar{h}$, and simplify other expressions correspondingly – in particular (163), (168) and (171) become (172)–(174). There is an alternative procedure, however, more in keeping with the spirit of the paper, since it amounts to letting $\bar{n}$ represent 'mean mean *draconic* motion' rather than 'mean mean *anomalistic* motion', the former being more practical for a near-circular orbit. This requires k_n to be given by (189) instead of (180), the derivation of the second-order component of the Kepler relation being as given in §7.9; then (following (190) and (278)) this relation becomes

$$\bar{n}^2\bar{a}^3 = \mu\{1+\tfrac{1}{2}\bar{K}(6-7\bar{f})+\tfrac{1}{24}\bar{K}^2\bar{f}(4-19\bar{f})\}, \tag{359}$$

and overall secular perturbations are represented by the formulae

$$\bar{\Omega} = \bar{\Omega}_0-\bar{K}\bar{n}\cos\bar{\imath}[1-\tfrac{5}{6}\bar{K}(3-4\bar{f})]\,t, \tag{360}$$

$$\bar{\omega} = \bar{\omega}_0+\tfrac{1}{2}\bar{K}\bar{n}(4-5\bar{f})\,t \tag{361}$$

and

$$\bar{U} = \bar{U}_0+\bar{n}t. \tag{362}$$

The $\bar{K}^2$ term in (360) comes from (227); it differs from the $\bar{K}^2$ term in (327) as a direct consequence of the change in k_n. As already mentioned, the variation in $\bar{\omega}$ may be suppressed completely, by adding extra terms into (172) and (173) as compensation.

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Note added in proof. In assigning 'preferred' values to the k-constants, in § 6.3, it appeared that the simplest formula for r_1' would be (163), with k_a given by (160) and the associated semi-major axis only slightly different from that in Kozai's paper. This is only true, however, if it is demanded that the mean of r_1' be $O(\bar{e})$, i.e. that $\bar{r}$ be an essentially unbiased mean. Equation (163) is marred by the internal factor $(\bar{r}/\bar{p})$ and the simplest formula for r_1' is given by using (179) for k_a, i.e. by using a' as $\bar{a}$. We then also need k_e, such that $4\bar{h}$ is replaced by $\frac{20}{3}\bar{h}$ in (161), (176) and (177), the reward being the simple expressions for both r_1' and u_1' that are inherent in (366) and (367) below. The use of a' makes $\bar{p}/\bar{r}$ a factor of a_1 (this applies to the analysis for an arbitrary J_n), and a' is naturally retained in the second-order analysis; this constrains k_{2a} to be $\frac{2}{9}(5-15\bar{f}+12\bar{f}^2)$, rather than as given by (211). Revised formulae for a_2, i_2 etc. follow from (209), (216) etc., the simplicity of (321) being preserved by taking $17-25\bar{f}$ for k_{2i}.

The effect of the changes in k_a and k_e on the most accurate second-order algorithm of § 10 is that (327), (337) and (357) must be replaced by

$$\dot{\bar{\Omega}}_2 = -\tfrac{1}{24}\bar{n}\cos\bar{i}\{4(1-\bar{f})+\bar{e}^2(4+5\bar{f})\}, \tag{363}$$

$$\dot{\bar{\omega}}_2 = \tfrac{1}{96}\bar{n}\{2(64-180\bar{f}+95\bar{f}^2)+\bar{e}^2(56-36\bar{f}-45\bar{f}^2)\} \tag{364}$$

and

$$\bar{n}^2\,\bar{a}^3 = \mu\{1-\tfrac{1}{24}\bar{K}^2\bar{q}^3(8-8\bar{f}-5\bar{f}^2)\}, \tag{365}$$

where $\bar{a} = a'$ and the k_{2n} associated with $\bar{n}$ is $-\frac{1}{24}(16-56\bar{f}+31\bar{f}^2)$ instead of (349). (Zero suffices have been omitted from (365), and the factor $\bar{q}^3$ has been included as it is known from (346) that this is the untruncated relation.) The combined revisions of (163), (168), (171), (297), (315) and (321) can now be expressed by

$$r' = \bar{r}+\tfrac{1}{6}\bar{K}\bar{a}\{(\bar{f}\bar{q}^2\cos 2\bar{u}-2\bar{h}) - \tfrac{1}{12}\bar{K}[\bar{f}^2\cos 4\bar{u}+8\bar{f}(6-7\bar{f})\cos 2\bar{u}+4(4-12\bar{f}-19\bar{f}^2)]\}, \tag{366}$$

$$u' = \bar{u}+\tfrac{1}{12}\bar{K}\{\bar{f}[\sin 2\bar{u}+4\bar{e}\sin(\bar{u}+\bar{\omega})]+\bar{h}\bar{e}[32\sin\bar{v}-9\bar{e}V(\bar{v},\bar{e})] - \tfrac{1}{6}\bar{K}\bar{f}[\bar{f}\sin 4\bar{u}-(23-26\bar{f})\sin 2\bar{u}]\} \tag{367}$$

and

$$c = \tfrac{1}{24}\bar{K}r\sin 2\bar{i}\{\bar{e}[16\sin(\bar{u}+\bar{v})-24\sin\bar{\omega}-9\bar{e}\,V(\bar{v},\bar{e})\cos\bar{u}]-2\bar{K}\bar{f}\sin 3\bar{u}\}. \tag{368}$$

Over short periods of time such that $\dot{\bar{\omega}}_2$ can be dropped, (358) is replaced by

$$\bar{n}^2\,\bar{a}^3 = \mu\{1+\tfrac{1}{6}\bar{K}^2(14-43\bar{f}+25\bar{f}^2)\}. \tag{369}$$

Finally, when $\bar{n}$ is defined on the basis of (362), equations (359) and (360) are replaced by

$$\bar{n}^2\,\bar{a}^3 = \mu\{1+\bar{K}(4-5\bar{f})+\tfrac{1}{12}\bar{K}^2(76-206\bar{f}+125\bar{f}^2)\} \tag{370}$$

and

$$\bar{\Omega} = \bar{\Omega}_0-\bar{K}\bar{n}\cos\bar{i}[1-\tfrac{1}{6}\bar{K}(11-14\bar{f})]t. \tag{371}$$

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